CHAPTER 10 INFINITE SEQUENCES AND SERIES

10.1 SEQUENCES

1. $a_1 = \frac{1-1}{1^2} = 0$, $a_2 = \frac{1-2}{2^2} = -\frac{1}{4}$, $a_3 = \frac{1-3}{3^2} = -\frac{2}{9}$, $a_4 = \frac{1-4}{4^2} = -\frac{3}{16}$

2. $a_1 = \frac{1}{1!} = 1$, $a_2 = \frac{1}{2!} = \frac{1}{2}$, $a_3 = \frac{1}{3!} = \frac{1}{6}$, $a_4 = \frac{1}{4!} = \frac{1}{24}$

3. $a_1 = \frac{(-1)^2}{2-1} = 1$, $a_2 = \frac{(-1)^3}{4-1} = -\frac{1}{3}$, $a_3 = \frac{(-1)^4}{6-1} = \frac{1}{5}$, $a_4 = \frac{(-1)^5}{8-1} = -\frac{1}{7}$

4. $a_1 = 2 + (-1)^1 = 1$, $a_2 = 2 + (-1)^2 = 3$, $a_3 = 2 + (-1)^3 = 1$, $a_4 = 2 + (-1)^4 = 3$

5. $a_1 = \frac{2}{2^2} = \frac{1}{2}$, $a_2 = \frac{2^2}{2^3} = \frac{1}{2}$, $a_3 = \frac{2^3}{2^4} = \frac{1}{2}$, $a_4 = \frac{2^4}{2^5} = \frac{1}{2}$

6. $a_1 = \frac{2-1}{2} = \frac{1}{2}$, $a_2 = \frac{2^2-1}{2^2} = \frac{3}{4}$, $a_3 = \frac{2^3-1}{2^3} = \frac{7}{8}$, $a_4 = \frac{2^4-1}{2^4} = \frac{15}{16}$

7. $a_1 = 1$, $a_2 = 1 + \frac{1}{2} = \frac{3}{2}$, $a_3 = \frac{3}{2} + \frac{1}{2^2} = \frac{7}{4}$, $a_4 = \frac{7}{4} + \frac{1}{2^3} = \frac{15}{8}$, $a_5 = \frac{15}{8} + \frac{1}{2^4} = \frac{31}{16}$, $a_6 = \frac{63}{32}$,
$a_7 = \frac{127}{64}$, $a_8 = \frac{255}{128}$, $a_9 = \frac{511}{256}$, $a_{10} = \frac{1023}{512}$

8. $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{\left(\frac{1}{2}\right)}{3} = \frac{1}{6}$, $a_4 = \frac{\left(\frac{1}{6}\right)}{4} = \frac{1}{24}$, $a_5 = \frac{\left(\frac{1}{24}\right)}{5} = \frac{1}{120}$, $a_6 = \frac{1}{720}$, $a_7 = \frac{1}{5040}$, $a_8 = \frac{1}{40{,}320}$,
$a_9 = \frac{1}{362{,}880}$, $a_{10} = \frac{1}{3{,}628{,}800}$

9. $a_1 = 2$, $a_2 = \frac{(-1)^2(2)}{2} = 1$, $a_3 = \frac{(-1)^3(1)}{2} = -\frac{1}{2}$, $a_4 = \frac{(-1)^4\left(-\frac{1}{2}\right)}{2} = -\frac{1}{4}$, $a_5 = \frac{(-1)^5\left(-\frac{1}{4}\right)}{2} = \frac{1}{8}$,
$a_6 = \frac{1}{16}$, $a_7 = -\frac{1}{32}$, $a_8 = -\frac{1}{64}$, $a_9 = \frac{1}{128}$, $a_{10} = \frac{1}{256}$

10. $a_1 = -2$, $a_2 = \frac{1\cdot(-2)}{2} = -1$, $a_3 = \frac{2\cdot(-1)}{3} = -\frac{2}{3}$, $a_4 = \frac{3\cdot\left(-\frac{2}{3}\right)}{4} = -\frac{1}{2}$, $a_5 = \frac{4\cdot\left(-\frac{1}{2}\right)}{5} = -\frac{2}{5}$, $a_6 = -\frac{1}{3}$,
$a_7 = -\frac{2}{7}$, $a_8 = -\frac{1}{4}$, $a_9 = -\frac{2}{9}$, $a_{10} = -\frac{1}{5}$

11. $a_1 = 1$, $a_2 = 1$, $a_3 = 1 + 1 = 2$, $a_4 = 2 + 1 = 3$, $a_5 = 3 + 2 = 5$, $a_6 = 8$, $a_7 = 13$, $a_8 = 21$, $a_9 = 34$, $a_{10} = 55$

12. $a_1 = 2$, $a_2 = -1$, $a_3 = -\frac{1}{2}$, $a_4 = \frac{\left(-\frac{1}{2}\right)}{-1} = \frac{1}{2}$, $a_5 = \frac{\left(\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)} = -1$, $a_6 = -2$, $a_7 = 2$, $a_8 = -1$, $a_9 = -\frac{1}{2}$, $a_{10} = \frac{1}{2}$

13. $a_n = (-1)^{n+1}$, $n = 1, 2, \ldots$

14. $a_n = (-1)^n$, $n = 1, 2, \ldots$

15. $a_n = (-1)^{n+1} n^2$, $n = 1, 2, \ldots$

16. $a_n = \frac{(-1)^{n+1}}{n^2}$, $n = 1, 2, \ldots$

17. $a_n = \frac{2^{n-1}}{3(n+2)}$, $n = 1, 2, \ldots$

18. $a_n = \frac{2n-5}{n(n+1)}$, $n = 1, 2, \ldots$

19. $a_n = n^2 - 1$, $n = 1, 2, \ldots$

20. $a_n = n - 4$, $n = 1, 2, \ldots$

21. $a_n = 4n - 3$, $n = 1, 2, \ldots$

22. $a_n = 4n - 2$, $n = 1, 2, \ldots$

23. $a_n = \frac{3n+2}{n!}$, $n = 1, 2, \ldots$

24. $a_n = \frac{n^3}{5^{n+1}}$, $n = 1, 2, \ldots$

25. $a_n = \frac{1+(-1)^{n+1}}{2}$, $n = 1, 2, \ldots$

26. $a_n = \frac{n - \frac{1}{2} + (-1)^n \left(\frac{1}{2}\right)}{2} = \left\lfloor \frac{n}{2} \right\rfloor$, $n = 1, 2, \ldots$

27. $\lim_{n\to\infty} 2 + (0.1)^n = 2 \Rightarrow$ converges (Theorem 5, #4)

28. $\lim_{n\to\infty} \frac{n+(-1)^n}{n} = \lim_{n\to\infty} 1 + \frac{(-1)^n}{n} = 1 \Rightarrow$ converges

29. $\lim_{n\to\infty} \frac{1-2n}{1+2n} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)-2}{\left(\frac{1}{n}\right)+2} = \lim_{n\to\infty} \frac{-2}{2} = -1 \Rightarrow$ converges

30. $\lim_{n\to\infty} \frac{2n+1}{1-3\sqrt{n}} = \lim_{n\to\infty} \frac{2\sqrt{n} + \left(\frac{1}{\sqrt{n}}\right)}{\left(\frac{1}{\sqrt{n}} - 3\right)} = -\infty \Rightarrow$ diverges

31. $\lim_{n\to\infty} \frac{1-5n^4}{n^4+8n^3} = \lim_{n\to\infty} \frac{\left(\frac{1}{n^4}\right)-5}{1+\left(\frac{8}{n}\right)} = -5 \Rightarrow$ converges

32. $\lim_{n\to\infty} \frac{n+3}{n^2+5n+6} = \lim_{n\to\infty} \frac{n+3}{(n+3)(n+2)} = \lim_{n\to\infty} \frac{1}{n+2} = 0 \Rightarrow$ converges

33. $\lim_{n\to\infty} \frac{n^2-2n+1}{n-1} = \lim_{n\to\infty} \frac{(n-1)(n-1)}{n-1} = \lim_{n\to\infty} (n-1) = \infty \Rightarrow$ diverges

34 $\lim_{n\to\infty} \frac{1-n^3}{70-4n^2} = \lim_{n\to\infty} \frac{\left(\frac{1}{n^2}\right)-n}{\left(\frac{70}{n^2}\right)-4} = \infty \Rightarrow$ diverges

35. $\lim_{n\to\infty} (1+(-1)^n)$ does not exist $\Rightarrow$ diverges

36. $\lim_{n\to\infty} (-1)^n \left(1 - \frac{1}{n}\right)$ does not exist $\Rightarrow$ diverges

37. $\lim_{n\to\infty} \left(\frac{n+1}{2n}\right)\left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2n}\right)\left(1 - \frac{1}{n}\right) = \frac{1}{2} \Rightarrow$ converges

38. $\lim_{n\to\infty} \left(2 - \frac{1}{2^n}\right)\left(3 + \frac{1}{2^n}\right) = 6 \Rightarrow$ converges

39. $\lim_{n\to\infty} \frac{(-1)^{n+1}}{2n-1} = 0 \Rightarrow$ converges

40. $\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = \lim_{n\to\infty} \frac{(-1)^n}{2^n} = 0 \Rightarrow$ converges

41. $\lim_{n\to\infty} \sqrt{\frac{2n}{n+1}} = \sqrt{\lim_{n\to\infty} \frac{2n}{n+1}} = \sqrt{\lim_{n\to\infty} \left(\frac{2}{1+\frac{1}{n}}\right)} = \sqrt{2} \Rightarrow$ converges

42. $\lim_{n\to\infty} \frac{1}{(0.9)^n} = \lim_{n\to\infty} \left(\frac{10}{9}\right)^n = \infty \Rightarrow$ diverges

43. $\lim_{n\to\infty} \sin\left(\frac{\pi}{2} + \frac{1}{n}\right) = \sin\left(\lim_{n\to\infty} \left(\frac{\pi}{2} + \frac{1}{n}\right)\right) = \sin\frac{\pi}{2} = 1 \Rightarrow$ converges

44. $\lim_{n\to\infty} n\pi\cos(n\pi) = \lim_{n\to\infty} (n\pi)(-1)^n$ does not exist $\Rightarrow$ diverges

45. $\lim_{n\to\infty} \frac{\sin n}{n} = 0$ because $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n} \Rightarrow$ converges by the Sandwich Theorem for sequences

46. $\lim_{n\to\infty} \frac{\sin^2 n}{2^n} = 0$ because $0 \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n} \Rightarrow$ converges by the Sandwich Theorem for sequences

47. $\lim_{n\to\infty} \frac{n}{2^n} = \lim_{n\to\infty} \frac{1}{2^n \ln 2} = 0 \Rightarrow$ converges (using l'Hôpital's rule)

48. $\lim_{n \to \infty} \frac{3^n}{n^3} = \lim_{n \to \infty} \frac{3^n \ln 3}{3n^2} = \lim_{n \to \infty} \frac{3^n (\ln 3)^2}{6n} = \lim_{n \to \infty} \frac{3^n (\ln 3)^3}{6} = \infty \Rightarrow$ diverges (using l'Hôpital's rule)

49. $\lim_{n \to \infty} \frac{\ln(n+1)}{\sqrt{n}} = \lim_{n \to \infty} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{2\sqrt{n}}\right)} = \lim_{n \to \infty} \frac{2\sqrt{n}}{n+1} = \lim_{n \to \infty} \frac{\left(\frac{2}{\sqrt{n}}\right)}{1+\left(\frac{1}{n}\right)} = 0 \Rightarrow$ converges

50. $\lim_{n \to \infty} \frac{\ln n}{\ln 2n} = \lim_{n \to \infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{2}{2n}\right)} = 1 \Rightarrow$ converges

51. $\lim_{n \to \infty} 8^{1/n} = 1 \Rightarrow$ converges (Theorem 5, #3)

52. $\lim_{n \to \infty} (0.03)^{1/n} = 1 \Rightarrow$ converges (Theorem 5, #3)

53. $\lim_{n \to \infty} \left(1 + \frac{7}{n}\right)^n = e^7 \Rightarrow$ converges (Theorem 5, #5)

54. $\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = \lim_{n \to \infty} \left[1 + \frac{(-1)}{n}\right]^n = e^{-1} \Rightarrow$ converges (Theorem 5, #5)

55. $\lim_{n \to \infty} \sqrt[n]{10n} = \lim_{n \to \infty} 10^{1/n} \cdot n^{1/n} = 1 \cdot 1 = 1 \Rightarrow$ converges (Theorem 5, #3 and #2)

56. $\lim_{n \to \infty} \sqrt[n]{n^2} = \lim_{n \to \infty} \left(\sqrt[n]{n}\right)^2 = 1^2 = 1 \Rightarrow$ converges (Theorem 5, #2)

57. $\lim_{n \to \infty} \left(\frac{3}{n}\right)^{1/n} = \frac{\lim_{n\to\infty} 3^{1/n}}{\lim_{n\to\infty} n^{1/n}} = \frac{1}{1} = 1 \Rightarrow$ converges (Theorem 5, #3 and #2)

58. $\lim_{n \to \infty} (n+4)^{1/(n+4)} = \lim_{x \to \infty} x^{1/x} = 1 \Rightarrow$ converges; (let $x = n + 4$, then use Theorem 5, #2)

59. $\lim_{n \to \infty} \frac{\ln n}{n^{1/n}} = \frac{\lim_{n\to\infty} \ln n}{\lim_{n\to\infty} n^{1/n}} = \frac{\infty}{1} = \infty \Rightarrow$ diverges (Theorem 5, #2)

60. $\lim_{n \to \infty} [\ln n - \ln(n+1)] = \lim_{n \to \infty} \ln\left(\frac{n}{n+1}\right) = \ln\left(\lim_{n \to \infty} \frac{n}{n+1}\right) = \ln 1 = 0 \Rightarrow$ converges

61. $\lim_{n \to \infty} \sqrt[n]{4^n n} = \lim_{n \to \infty} 4\sqrt[n]{n} = 4 \cdot 1 = 4 \Rightarrow$ converges (Theorem 5, #2)

62. $\lim_{n \to \infty} \sqrt[n]{3^{2n+1}} = \lim_{n \to \infty} 3^{2+(1/n)} = \lim_{n \to \infty} 3^2 \cdot 3^{1/n} = 9 \cdot 1 = 9 \Rightarrow$ converges (Theorem 5, #3)

63. $\lim_{n \to \infty} \frac{n!}{n^n} = \lim_{n \to \infty} \frac{1 \cdot 2 \cdot 3 \cdots (n-1)(n)}{n \cdot n \cdot n \cdots n \cdot n} \le \lim_{n \to \infty} \left(\frac{1}{n}\right) = 0$ and $\frac{n!}{n^n} \ge 0 \Rightarrow \lim_{n \to \infty} \frac{n!}{n^n} = 0 \Rightarrow$ converges

64. $\lim_{n \to \infty} \frac{(-4)^n}{n!} = 0 \Rightarrow$ converges (Theorem 5, #6)

65. $\lim_{n \to \infty} \frac{n!}{10^{6n}} = \lim_{n \to \infty} \frac{1}{\left(\frac{(10^6)^n}{n!}\right)} = \infty \Rightarrow$ diverges (Theorem 5, #6)

66. $\lim_{n \to \infty} \frac{n!}{2^n 3^n} = \lim_{n \to \infty} \frac{1}{\left(\frac{6^n}{n!}\right)} = \infty \Rightarrow$ diverges (Theorem 5, #6)

67. $\lim_{n \to \infty} \left(\frac{1}{n}\right)^{1/(\ln n)} = \lim_{n \to \infty} \exp\left(\frac{1}{\ln n} \ln\left(\frac{1}{n}\right)\right) = \lim_{n \to \infty} \exp\left(\frac{\ln 1 - \ln n}{\ln n}\right) = e^{-1} \Rightarrow$ converges

68. $\lim\limits_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = \ln\left(\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right) = \ln e = 1 \Rightarrow$ converges (Theorem 5, #5)

69. $\lim\limits_{n\to\infty}\left(\frac{3n+1}{3n-1}\right)^n = \lim\limits_{n\to\infty}\exp\left(n\ln\left(\frac{3n+1}{3n-1}\right)\right) = \lim\limits_{n\to\infty}\exp\left(\frac{\ln(3n+1)-\ln(3n-1)}{\frac{1}{n}}\right)$
$= \lim\limits_{n\to\infty}\exp\left(\frac{\frac{3}{3n+1}-\frac{3}{3n-1}}{\left(-\frac{1}{n^2}\right)}\right) = \lim\limits_{n\to\infty}\exp\left(\frac{6n^2}{(3n+1)(3n-1)}\right) = \exp\left(\frac{6}{9}\right) = e^{2/3} \Rightarrow$ converges

70. $\lim\limits_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim\limits_{n\to\infty}\exp\left(n\ln\left(\frac{n}{n+1}\right)\right) = \lim\limits_{n\to\infty}\exp\left(\frac{\ln n-\ln(n+1)}{\left(\frac{1}{n}\right)}\right) = \lim\limits_{n\to\infty}\exp\left(\frac{\frac{1}{n}-\frac{1}{n+1}}{\left(-\frac{1}{n^2}\right)}\right)$
$= \lim\limits_{n\to\infty}\exp\left(-\frac{n^2}{n(n+1)}\right) = e^{-1} \Rightarrow$ converges

71. $\lim\limits_{n\to\infty}\left(\frac{x^n}{2n+1}\right)^{1/n} = \lim\limits_{n\to\infty} x\left(\frac{1}{2n+1}\right)^{1/n} = x\lim\limits_{n\to\infty}\exp\left(\frac{1}{n}\ln\left(\frac{1}{2n+1}\right)\right) = x\lim\limits_{n\to\infty}\exp\left(\frac{-\ln(2n+1)}{n}\right)$
$= x\lim\limits_{n\to\infty}\exp\left(\frac{-2}{2n+1}\right) = xe^0 = x,\ x > 0 \Rightarrow$ converges

72. $\lim\limits_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n = \lim\limits_{n\to\infty}\exp\left(n\ln\left(1-\frac{1}{n^2}\right)\right) = \lim\limits_{n\to\infty}\exp\left(\frac{\ln\left(1-\frac{1}{n^2}\right)}{\left(\frac{1}{n}\right)}\right) = \lim\limits_{n\to\infty}\exp\left[\frac{\left(\frac{2}{n^3}\right)/\left(1-\frac{1}{n^2}\right)}{\left(-\frac{1}{n^2}\right)}\right]$
$= \lim\limits_{n\to\infty}\exp\left(\frac{-2n}{n^2-1}\right) = e^0 = 1 \Rightarrow$ converges

73. $\lim\limits_{n\to\infty}\frac{3^n\cdot 6^n}{2^{-n}\cdot n!} = \lim\limits_{n\to\infty}\frac{36^n}{n!} = 0 \Rightarrow$ converges (Theorem 5, #6)

74. $\lim\limits_{n\to\infty}\frac{\left(\frac{10}{11}\right)^n}{\left(\frac{9}{10}\right)^n+\left(\frac{11}{12}\right)^n} = \lim\limits_{n\to\infty}\frac{\left(\frac{12}{11}\right)^n\left(\frac{10}{11}\right)^n}{\left(\frac{12}{11}\right)^n\left(\frac{9}{10}\right)^n+\left(\frac{12}{11}\right)^n\left(\frac{11}{12}\right)^n} = \lim\limits_{n\to\infty}\frac{\left(\frac{120}{121}\right)^n}{\left(\frac{108}{110}\right)^n+1} = 0 \Rightarrow$ converges
(Theorem 5, #4)

75. $\lim\limits_{n\to\infty}\tanh n = \lim\limits_{n\to\infty}\frac{e^n-e^{-n}}{e^n+e^{-n}} = \lim\limits_{n\to\infty}\frac{e^{2n}-1}{e^{2n}+1} = \lim\limits_{n\to\infty}\frac{2e^{2n}}{2e^{2n}} = \lim\limits_{n\to\infty} 1 = 1 \Rightarrow$ converges

76. $\lim\limits_{n\to\infty}\sinh(\ln n) = \lim\limits_{n\to\infty}\frac{e^{\ln n}-e^{-\ln n}}{2} = \lim\limits_{n\to\infty}\frac{n-\left(\frac{1}{n}\right)}{2} = \infty \Rightarrow$ diverges

77. $\lim\limits_{n\to\infty}\frac{n^2\sin\left(\frac{1}{n}\right)}{2n-1} = \lim\limits_{n\to\infty}\frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{2}{n}-\frac{1}{n^2}\right)} = \lim\limits_{n\to\infty}\frac{-\left(\cos\left(\frac{1}{n}\right)\right)\left(\frac{1}{n^2}\right)}{\left(-\frac{2}{n^2}+\frac{2}{n^3}\right)} = \lim\limits_{n\to\infty}\frac{-\cos\left(\frac{1}{n}\right)}{-2+\left(\frac{2}{n}\right)} = \frac{1}{2} \Rightarrow$ converges

78. $\lim\limits_{n\to\infty} n\left(1-\cos\frac{1}{n}\right) = \lim\limits_{n\to\infty}\frac{\left(1-\cos\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim\limits_{n\to\infty}\frac{\left[\sin\left(\frac{1}{n}\right)\right]\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim\limits_{n\to\infty}\sin\left(\frac{1}{n}\right) = 0 \Rightarrow$ converges

79. $\lim\limits_{n\to\infty}\sqrt{n}\sin\left(\frac{1}{\sqrt{n}}\right) = \lim\limits_{n\to\infty}\frac{\sin\left(\frac{1}{\sqrt{n}}\right)}{\frac{1}{\sqrt{n}}} = \lim\limits_{n\to\infty}\frac{\cos\left(\frac{1}{\sqrt{n}}\right)\left(-\frac{1}{2n^{3/2}}\right)}{-\frac{1}{2n^{3/2}}} = \lim\limits_{n\to\infty}\cos\left(\frac{1}{\sqrt{n}}\right) = \cos 0 = 1 \Rightarrow$ converges

80. $\lim\limits_{n\to\infty}(3^n+5^n)^{1/n} = \lim\limits_{n\to\infty}\exp\left[\ln(3^n+5^n)^{1/n}\right] = \lim\limits_{n\to\infty}\exp\left[\frac{\ln(3^n+5^n)}{n}\right] = \lim\limits_{n\to\infty}\exp\left[\frac{\frac{3^n\ln 3+5^n\ln 5}{3^n+5^n}}{1}\right]$
$= \lim\limits_{n\to\infty}\exp\left[\frac{\left(\frac{3^n}{5^n}\right)\ln 3+\ln 5}{\left(\frac{3^n}{5^n}\right)+1}\right] = \lim\limits_{n\to\infty}\exp\left[\frac{\left(\frac{3}{5}\right)^n\ln 3+\ln 5}{\left(\frac{3}{5}\right)^n+1}\right] = \exp(\ln 5) = 5$

81. $\lim\limits_{n\to\infty}\tan^{-1} n = \frac{\pi}{2} \Rightarrow$ converges

82. $\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}}\tan^{-1} n = 0\cdot\frac{\pi}{2} = 0 \Rightarrow$ converges

83. $\lim\limits_{n\to\infty} \left(\frac{1}{3}\right)^n + \frac{1}{\sqrt{2^n}} = \lim\limits_{n\to\infty} \left(\left(\frac{1}{3}\right)^n + \left(\frac{1}{\sqrt{2}}\right)^n\right) = 0 \Rightarrow$ converges (Theorem 5, #4)

84. $\lim\limits_{n\to\infty} \sqrt[n]{n^2+n} = \lim\limits_{n\to\infty} \exp\left[\frac{\ln(n^2+n)}{n}\right] = \lim\limits_{n\to\infty} \exp\left(\frac{2n+1}{n^2+n}\right) = e^0 = 1 \Rightarrow$ converges

85. $\lim\limits_{n\to\infty} \frac{(\ln n)^{200}}{n} = \lim\limits_{n\to\infty} \frac{200(\ln n)^{199}}{n} = \lim\limits_{n\to\infty} \frac{200\cdot 199(\ln n)^{198}}{n} = \ldots = \lim\limits_{n\to\infty} \frac{200!}{n} = 0 \Rightarrow$ converges

86. $\lim\limits_{n\to\infty} \frac{(\ln n)^5}{\sqrt{n}} = \lim\limits_{n\to\infty} \left[\frac{\left(\frac{5(\ln n)^4}{n}\right)}{\left(\frac{1}{2\sqrt{n}}\right)}\right] = \lim\limits_{n\to\infty} \frac{10(\ln n)^4}{\sqrt{n}} = \lim\limits_{n\to\infty} \frac{80(\ln n)^3}{\sqrt{n}} = \ldots = \lim\limits_{n\to\infty} \frac{3840}{\sqrt{n}} = 0 \Rightarrow$ converges

87. $\lim\limits_{n\to\infty} \left(n - \sqrt{n^2-n}\right) = \lim\limits_{n\to\infty} \left(n - \sqrt{n^2-n}\right)\left(\frac{n+\sqrt{n^2-n}}{n+\sqrt{n^2-n}}\right) = \lim\limits_{n\to\infty} \frac{n}{n+\sqrt{n^2-n}} = \lim\limits_{n\to\infty} \frac{1}{1+\sqrt{1-\frac{1}{n}}}$
$= \frac{1}{2} \Rightarrow$ converges

88. $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}} = \lim\limits_{n\to\infty} \left(\frac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}}\right)\left(\frac{\sqrt{n^2-1}+\sqrt{n^2+n}}{\sqrt{n^2-1}+\sqrt{n^2+n}}\right) = \lim\limits_{n\to\infty} \frac{\sqrt{n^2-1}+\sqrt{n^2+n}}{-1-n}$
$= \lim\limits_{n\to\infty} \frac{\sqrt{1-\frac{1}{n^2}}+\sqrt{1+\frac{1}{n}}}{\left(-\frac{1}{n}-1\right)} = -2 \Rightarrow$ converges

89. $\lim\limits_{n\to\infty} \frac{1}{n}\int_1^n \frac{1}{x}\,dx = \lim\limits_{n\to\infty} \frac{\ln n}{n} = \lim\limits_{n\to\infty} \frac{1}{n} = 0 \Rightarrow$ converges (Theorem 5, #1)

90. $\lim\limits_{n\to\infty} \int_1^n \frac{1}{x^p}\,dx = \lim\limits_{n\to\infty} \left[\frac{1}{1-p}\,\frac{1}{x^{p-1}}\right]_1^n = \lim\limits_{n\to\infty} \frac{1}{1-p}\left(\frac{1}{n^{p-1}}-1\right) = \frac{1}{p-1}$ if $p > 1 \Rightarrow$ converges

91. Since a_n converges $\Rightarrow \lim\limits_{n\to\infty} a_n = L \Rightarrow \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \frac{72}{1+a_n} \Rightarrow L = \frac{72}{1+L} \Rightarrow L(1+L) = 72 \Rightarrow L^2 + L - 72 = 0$
$\Rightarrow L = -9$ or $L = 8$; since $a_n > 0$ for $n \geq 1 \Rightarrow L = 8$

92. Since a_n converges $\Rightarrow \lim\limits_{n\to\infty} a_n = L \Rightarrow \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \frac{a_n+6}{a_n+2} \Rightarrow L = \frac{L+6}{L+2} \Rightarrow L(L+2) = L+6 \Rightarrow L^2 + L - 6 = 0$
$\Rightarrow L = -3$ or $L = 2$; since $a_n > 0$ for $n \geq 2 \Rightarrow L = 2$

93. Since a_n converges $\Rightarrow \lim\limits_{n\to\infty} a_n = L \Rightarrow \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \sqrt{8+2a_n} \Rightarrow L = \sqrt{8+2L} \Rightarrow L^2 - 2L - 8 = 0 \Rightarrow L = -2$
or $L = 4$; since $a_n > 0$ for $n \geq 3 \Rightarrow L = 4$

94. Since a_n converges $\Rightarrow \lim\limits_{n\to\infty} a_n = L \Rightarrow \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \sqrt{8+2a_n} \Rightarrow L = \sqrt{8+2L} \Rightarrow L^2 - 2L - 8 = 0 \Rightarrow L = -2$
or $L = 4$; since $a_n > 0$ for $n \geq 2 \Rightarrow L = 4$

95. Since a_n converges $\Rightarrow \lim\limits_{n\to\infty} a_n = L \Rightarrow \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \sqrt{5a_n} \Rightarrow L = \sqrt{5L} \Rightarrow L^2 - 5L = 0 \Rightarrow L = 0$ or $L = 5$; since
$a_n > 0$ for $n \geq 1 \Rightarrow L = 5$

96. Since a_n converges $\Rightarrow \lim\limits_{n\to\infty} a_n = L \Rightarrow \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \left(12 - \sqrt{a_n}\right) \Rightarrow L = \left(12 - \sqrt{L}\right) \Rightarrow L^2 - 25L + 144 = 0$
$\Rightarrow L = 9$ or $L = 16$; since $12 - \sqrt{a_n} < 12$ for $n \geq 1 \Rightarrow L = 9$

97. $a_{n+1} = 2 + \frac{1}{a_n}$, $n \geq 1$, $a_1 = 2$. Since a_n converges $\Rightarrow \lim\limits_{n\to\infty} a_n = L \Rightarrow \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \left(2 + \frac{1}{a_n}\right) \Rightarrow L = 2 + \frac{1}{L}$
$\Rightarrow L^2 - 2L - 1 = 0 \Rightarrow L = 1 \pm \sqrt{2}$; since $a_n > 0$ for $n \geq 1 \Rightarrow L = 1 + \sqrt{2}$

98. $a_{n+1} = \sqrt{1 + a_n}$, $n \ge 1$, $a_1 = \sqrt{1}$. Since a_n converges $\Rightarrow \lim_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{1+a_n} \Rightarrow L = \sqrt{1+L}$ $\Rightarrow L^2 - L - 1 = 0 \Rightarrow L = \frac{1\pm\sqrt{5}}{2}$; since $a_n > 0$ for $n \ge 1 \Rightarrow L = \frac{1+\sqrt{5}}{2}$

99. $1, 1, 2, 4, 8, 16, 32, \ldots = 1, 2^0, 2^1, 2^2, 2^3, 2^4, 2^5, \ldots \Rightarrow x_1 = 1$ and $x_n = 2^{n-2}$ for $n \ge 2$

100. (a) $1^2 - 2(1)^2 = -1$, $3^2 - 2(2)^2 = 1$; let $f(a,b) = (a+2b)^2 - 2(a+b)^2 = a^2 + 4ab + 4b^2 - 2a^2 - 4ab - 2b^2$ $= 2b^2 - a^2$; $a^2 - 2b^2 = -1 \Rightarrow f(a,b) = 2b^2 - a^2 = 1$; $a^2 - 2b^2 = 1 \Rightarrow f(a,b) = 2b^2 - a^2 = -1$

(b) $r_n^2 - 2 = \left(\frac{a+2b}{a+b}\right)^2 - 2 = \frac{a^2+4ab+4b^2-2a^2-4ab-2b^2}{(a+b)^2} = \frac{-(a^2-2b^2)}{(a+b)^2} = \frac{\pm 1}{y_n^2} \Rightarrow r_n = \sqrt{2 \pm \left(\frac{1}{y_n}\right)^2}$

In the first and second fractions, $y_n \ge n$. Let $\frac{a}{b}$ represent the $(n-1)$th fraction where $\frac{a}{b} \ge 1$ and $b \ge n-1$ for n a positive integer ≥ 3. Now the nth fraction is $\frac{a+2b}{a+b}$ and $a + b \ge 2b \ge 2n - 2 \ge n \Rightarrow y_n \ge n$. Thus, $\lim_{n\to\infty} r_n = \sqrt{2}$.

101. (a) $f(x) = x^2 - 2$; the sequence converges to $1.414213562 \approx \sqrt{2}$

(b) $f(x) = \tan(x) - 1$; the sequence converges to $0.7853981635 \approx \frac{\pi}{4}$

(c) $f(x) = e^x$; the sequence $1, 0, -1, -2, -3, -4, -5, \ldots$ diverges

102. (a) $\lim_{n\to\infty} nf\left(\frac{1}{n}\right) = \lim_{\Delta x\to 0^+} \frac{f(\Delta x)}{\Delta x} = \lim_{\Delta x\to 0^+} \frac{f(0+\Delta x)-f(0)}{\Delta x} = f'(0)$, where $\Delta x = \frac{1}{n}$

(b) $\lim_{n\to\infty} n\tan^{-1}\left(\frac{1}{n}\right) = f'(0) = \frac{1}{1+0^2} = 1$, $f(x) = \tan^{-1} x$

(c) $\lim_{n\to\infty} n\left(e^{1/n} - 1\right) = f'(0) = e^0 = 1$, $f(x) = e^x - 1$

(d) $\lim_{n\to\infty} n\ln\left(1 + \frac{2}{n}\right) = f'(0) = \frac{2}{1+2(0)} = 2$, $f(x) = \ln(1+2x)$

103. (a) If $a = 2n+1$, then $b = \left\lfloor \frac{a^2}{2} \right\rfloor = \left\lfloor \frac{4n^2+4n+1}{2} \right\rfloor = \left\lfloor 2n^2 + 2n + \frac{1}{2} \right\rfloor = 2n^2 + 2n$, $c = \left\lceil \frac{a^2}{2} \right\rceil = \left\lceil 2n^2 + 2n + \frac{1}{2} \right\rceil$ $= 2n^2 + 2n + 1$ and $a^2 + b^2 = (2n+1)^2 + \left(2n^2+2n\right)^2 = 4n^2 + 4n + 1 + 4n^4 + 8n^3 + 4n^2$ $= 4n^4 + 8n^3 + 8n^2 + 4n + 1 = \left(2n^2 + 2n + 1\right)^2 = c^2$.

(b) $\lim_{a\to\infty} \frac{\left\lfloor \frac{a^2}{2} \right\rfloor}{\left\lceil \frac{a^2}{2} \right\rceil} = \lim_{a\to\infty} \frac{2n^2+2n}{2n^2+2n+1} = 1$ or $\lim_{a\to\infty} \frac{\left\lfloor \frac{a^2}{2} \right\rfloor}{\left\lceil \frac{a^2}{2} \right\rceil} = \lim_{a\to\infty} \sin\theta = \lim_{\theta\to\pi/2} \sin\theta = 1$

104. (a) $\lim_{n\to\infty} (2n\pi)^{1/(2n)} = \lim_{n\to\infty} \exp\left(\frac{\ln 2n\pi}{2n}\right) = \lim_{n\to\infty} \exp\left(\frac{\left(\frac{2\pi}{2n\pi}\right)}{2}\right) = \lim_{n\to\infty} \exp\left(\frac{1}{2n}\right) = e^0 = 1$;

$n! \approx \left(\frac{n}{e}\right) \sqrt[n]{2n\pi}$, Stirlings approximation $\Rightarrow \sqrt[n]{n!} \approx \left(\frac{n}{e}\right)(2n\pi)^{1/(2n)} \approx \frac{n}{e}$ for large values of n

(b)

n	$\sqrt[n]{n!}$	$\frac{n}{e}$
40	15.76852702	14.71517765
50	19.48325423	18.39397206
60	23.19189561	22.07276647

105. (a) $\lim_{n\to\infty} \frac{\ln n}{n^c} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{cn^{c-1}} = \lim_{n\to\infty} \frac{1}{cn^c} = 0$

(b) For all $\epsilon > 0$, there exists an $N = e^{-(\ln\epsilon)/c}$ such that $n > e^{-(\ln\epsilon)/c} \Rightarrow \ln n > -\frac{\ln\epsilon}{c} \Rightarrow \ln n^c > \ln\left(\frac{1}{\epsilon}\right)$ $\Rightarrow n^c > \frac{1}{\epsilon} \Rightarrow \frac{1}{n^c} < \epsilon \Rightarrow \left|\frac{1}{n^c} - 0\right| < \epsilon \Rightarrow \lim_{n\to\infty} \frac{1}{n^c} = 0$

106. Let $\{a_n\}$ and $\{b_n\}$ be sequences both converging to L. Define $\{c_n\}$ by $c_{2n} = b_n$ and $c_{2n-1} = a_n$, where $n = 1, 2, 3, \ldots$. For all $\epsilon > 0$ there exists N_1 such that when $n > N_1$ then $|a_n - L| < \epsilon$ and there exists N_2 such that when $n > N_2$ then $|b_n - L| < \epsilon$. If $n > 1 + 2\max\{N_1, N_2\}$, then $|c_n - L| < \epsilon$, so $\{c_n\}$ converges to L.

107. $\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \exp\left(\frac{1}{n}\ln n\right) = \lim_{n\to\infty} \exp\left(\frac{1}{n}\right) = e^0 = 1$

108. $\lim_{n\to\infty} x^{1/n} = \lim_{n\to\infty} \exp\left(\frac{1}{n}\ln x\right) = e^0 = 1$, because x remains fixed while n gets large

109. Assume the hypotheses of the theorem and let ϵ be a positive number. For all ϵ there exists a N_1 such that when $n > N_1$ then $|a_n - L| < \epsilon \Rightarrow -\epsilon < a_n - L < \epsilon \Rightarrow L - \epsilon < a_n$, and there exists a N_2 such that when $n > N_2$ then $|c_n - L| < \epsilon \Rightarrow -\epsilon < c_n - L < \epsilon \Rightarrow c_n < L + \epsilon$. If $n > \max\{N_1, N_2\}$, then $L - \epsilon < a_n \le b_n \le c_n < L + \epsilon \Rightarrow |b_n - L| < \epsilon \Rightarrow \lim_{n\to\infty} b_n = L$.

110. Let $\epsilon > 0$. We have f continuous at $L \Rightarrow$ there exists δ so that $|x - L| < \delta \Rightarrow |f(x) - f(L)| < \epsilon$. Also, $a_n \to L \Rightarrow$ there exists N so that for $n > N$ $|a_n - L| < \delta$. Thus for $n > N$, $|f(a_n) - f(L)| < \epsilon \Rightarrow f(a_n) \to f(L)$.

111. $a_{n+1} \ge a_n \Rightarrow \frac{3(n+1)+1}{(n+1)+1} > \frac{3n+1}{n+1} \Rightarrow \frac{3n+4}{n+2} > \frac{3n+1}{n+1} \Rightarrow 3n^2 + 3n + 4n + 4 > 3n^2 + 6n + n + 2$
$\Rightarrow 4 > 2$; the steps are reversible so the sequence is nondecreasing; $\frac{3n+1}{n+1} < 3 \Rightarrow 3n + 1 < 3n + 3$
$\Rightarrow 1 < 3$; the steps are reversible so the sequence is bounded above by 3

112. $a_{n+1} \ge a_n \Rightarrow \frac{(2(n+1)+3)!}{((n+1)+1)!} > \frac{(2n+3)!}{(n+1)!} \Rightarrow \frac{(2n+5)!}{(n+2)!} > \frac{(2n+3)!}{(n+1)!} \Rightarrow \frac{(2n+5)!}{(2n+3)!} > \frac{(n+2)!}{(n+1)!}$
$\Rightarrow (2n+5)(2n+4) > n + 2$; the steps are reversible so the sequence is nondecreasing; the sequence is not bounded since $\frac{(2n+3)!}{(n+1)!} = (2n+3)(2n+2)\cdots(n+2)$ can become as large as we please

113. $a_{n+1} \le a_n \Rightarrow \frac{2^{n+1}3^{n+1}}{(n+1)!} \le \frac{2^n 3^n}{n!} \Rightarrow \frac{2^{n+1}3^{n+1}}{2^n 3^n} \le \frac{(n+1)!}{n!} \Rightarrow 2 \cdot 3 \le n + 1$ which is true for $n \ge 5$; the steps are reversible so the sequence is decreasing after a_5, but it is not nondecreasing for all its terms; $a_1 = 6$, $a_2 = 18$, $a_3 = 36$, $a_4 = 54$, $a_5 = \frac{324}{5} = 64.8 \Rightarrow$ the sequence is bounded from above by 64.8

114. $a_{n+1} \ge a_n \Rightarrow 2 - \frac{2}{n+1} - \frac{1}{2^{n+1}} \ge 2 - \frac{2}{n} - \frac{1}{2^n} \Rightarrow \frac{2}{n} - \frac{2}{n+1} \ge \frac{1}{2^{n+1}} - \frac{1}{2^n} \Rightarrow \frac{2}{n(n+1)} \ge -\frac{1}{2^{n+1}}$; the steps are reversible so the sequence is nondecreasing; $2 - \frac{2}{n} - \frac{1}{2^n} \le 2 \Rightarrow$ the sequence is bounded from above

115. $a_n = 1 - \frac{1}{n}$ converges because $\frac{1}{n} \to 0$ by Example 1; also it is a nondecreasing sequence bounded above by 1

116. $a_n = n - \frac{1}{n}$ diverges because $n \to \infty$ and $\frac{1}{n} \to 0$ by Example 1, so the sequence is unbounded

117. $a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$ and $0 < \frac{1}{2^n} < \frac{1}{n}$; since $\frac{1}{n} \to 0$ (by Example 1) $\Rightarrow \frac{1}{2^n} \to 0$, the sequence converges; also it is a nondecreasing sequence bounded above by 1

118. $a_n = \frac{2^n - 1}{3^n} = \left(\frac{2}{3}\right)^n - \frac{1}{3^n}$; the sequence converges to 0 by Theorem 5, #4

119. $a_n = ((-1)^n + 1)\left(\frac{n+1}{n}\right)$ diverges because $a_n = 0$ for n odd, while for n even $a_n = 2\left(1 + \frac{1}{n}\right)$ converges to 2; it diverges by definition of divergence

120. $x_n = \max\{\cos 1, \cos 2, \cos 3, \ldots, \cos n\}$ and $x_{n+1} = \max\{\cos 1, \cos 2, \cos 3, \ldots, \cos(n+1)\} \ge x_n$ with $x_n \le 1$ so the sequence is nondecreasing and bounded above by 1 $\Rightarrow$ the sequence converges.

121. $a_n \ge a_{n+1} \Leftrightarrow \frac{1+\sqrt{2n}}{\sqrt{n}} \ge \frac{1+\sqrt{2(n+1)}}{\sqrt{n+1}} \Leftrightarrow \sqrt{n+1} + \sqrt{2n^2+2n} \ge \sqrt{n} + \sqrt{2n^2+2n} \Leftrightarrow \sqrt{n+1} \ge \sqrt{n}$ and $\frac{1+\sqrt{2n}}{\sqrt{n}} \ge \sqrt{2}$; thus the sequence is nonincreasing and bounded below by $\sqrt{2} \Rightarrow$ it converges

122. $a_n \ge a_{n+1} \Leftrightarrow \frac{n+1}{n} \ge \frac{(n+1)+1}{n+1} \Leftrightarrow n^2 + 2n + 1 \ge n^2 + 2n \Leftrightarrow 1 \ge 0$ and $\frac{n+1}{n} \ge 1$; thus the sequence is nonincreasing and bounded below by 1 $\Rightarrow$ it converges

123. $\frac{4^{n+1}+3^n}{4^n} = 4 + \left(\frac{3}{4}\right)^n$ so $a_n \ge a_{n+1} \Leftrightarrow 4 + \left(\frac{3}{4}\right)^n \ge 4 + \left(\frac{3}{4}\right)^{n+1} \Leftrightarrow \left(\frac{3}{4}\right)^n \ge \left(\frac{3}{4}\right)^{n+1} \Leftrightarrow 1 \ge \frac{3}{4}$ and $4 + \left(\frac{3}{4}\right)^n \ge 4$; thus the sequence is nonincreasing and bounded below by 4 $\Rightarrow$ it converges

124. $a_1 = 1$, $a_2 = 2 - 3$, $a_3 = 2(2-3) - 3 = 2^2 - (2^2 - 1) \cdot 3$, $a_4 = 2\left(2^2 - (2^2 - 1) \cdot 3\right) - 3 = 2^3 - (2^3 - 1)\,3$, $a_5 = 2\left[2^3 - (2^3 - 1)\,3\right] - 3 = 2^4 - (2^4 - 1)\,3, \ldots, a_n = 2^{n-1} - (2^{n-1} - 1)\,3 = 2^{n-1} - 3 \cdot 2^{n-1} + 3$ $= 2^{n-1}(1-3) + 3 = -2^n + 3$; $a_n \ge a_{n+1} \Leftrightarrow -2^n + 3 \ge -2^{n+1} + 3 \Leftrightarrow -2^n \ge -2^{n+1} \Leftrightarrow 1 \le 2$ so the sequence is nonincreasing but not bounded below and therefore diverges

125. Let $0 < M < 1$ and let N be an integer greater than $\frac{M}{1-M}$. Then $n > N \Rightarrow n > \frac{M}{1-M} \Rightarrow n - nM > M$ $\Rightarrow n > M + nM \Rightarrow n > M(n+1) \Rightarrow \frac{n}{n+1} > M$.

126. Since M_1 is a least upper bound and M_2 is an upper bound, $M_1 \le M_2$. Since M_2 is a least upper bound and M_1 is an upper bound, $M_2 \le M_1$. We conclude that $M_1 = M_2$ so the least upper bound is unique.

127. The sequence $a_n = 1 + \frac{(-1)^n}{2}$ is the sequence $\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$. This sequence is bounded above by $\frac{3}{2}$, but it clearly does not converge, by definition of convergence.

128. Let L be the limit of the convergent sequence $\{a_n\}$. Then by definition of convergence, for $\frac{\epsilon}{2}$ there corresponds an N such that for all m and n, $m > N \Rightarrow |a_m - L| < \frac{\epsilon}{2}$ and $n > N \Rightarrow |a_n - L| < \frac{\epsilon}{2}$. Now $|a_m - a_n| = |a_m - L + L - a_n| \le |a_m - L| + |L - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ whenever $m > N$ and $n > N$.

129. Given an $\epsilon > 0$, by definition of convergence there corresponds an N such that for all $n > N$, $|L_1 - a_n| < \epsilon$ and $|L_2 - a_n| < \epsilon$. Now $|L_2 - L_1| = |L_2 - a_n + a_n - L_1| \le |L_2 - a_n| + |a_n - L_1| < \epsilon + \epsilon = 2\epsilon$. $|L_2 - L_1| < 2\epsilon$ says that the difference between two fixed values is smaller than any positive number 2ϵ. The only nonnegative number smaller than every positive number is 0, so $|L_1 - L_2| = 0$ or $L_1 = L_2$.

130. Let k(n) and i(n) be two order-preserving functions whose domains are the set of positive integers and whose ranges are a subset of the positive integers. Consider the two subsequences $a_{k(n)}$ and $a_{i(n)}$, where $a_{k(n)} \to L_1$, $a_{i(n)} \to L_2$ and $L_1 \ne L_2$. Thus $\left|a_{k(n)} - a_{i(n)}\right| \to |L_1 - L_2| > 0$. So there does not exist N such that for all $m, n > N$ $\Rightarrow |a_m - a_n| < \epsilon$. So by Exercise 128, the sequence $\{a_n\}$ is not convergent and hence diverges.

131. $a_{2k} \to L \Leftrightarrow$ given an $\epsilon > 0$ there corresponds an N_1 such that $\left[2k > N_1 \Rightarrow |a_{2k} - L| < \epsilon\right]$. Similarly, $a_{2k+1} \to L \Leftrightarrow \left[2k + 1 > N_2 \Rightarrow |a_{2k+1} - L| < \epsilon\right]$. Let $N = \max\{N_1, N_2\}$. Then $n > N \Rightarrow |a_n - L| < \epsilon$ whether n is even or odd, and hence $a_n \to L$.

132. Assume $a_n \to 0$. This implies that given an $\epsilon > 0$ there corresponds an N such that $n > N \Rightarrow |a_n - 0| < \epsilon$ $\Rightarrow |a_n| < \epsilon \Rightarrow ||a_n|| < \epsilon \Rightarrow ||a_n| - 0| < \epsilon \Rightarrow |a_n| \to 0$. On the other hand, assume $|a_n| \to 0$. This implies that given an $\epsilon > 0$ there corresponds an N such that for $n > N$, $||a_n| - 0| < \epsilon \Rightarrow ||a_n|| < \epsilon \Rightarrow |a_n| < \epsilon$ $\Rightarrow |a_n - 0| < \epsilon \Rightarrow a_n \to 0$.

133. (a) $f(x) = x^2 - a \Rightarrow f'(x) = 2x \Rightarrow x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} \Rightarrow x_{n+1} = \frac{2x_n^2 - (x_n^2 - a)}{2x_n} = \frac{x_n^2 + a}{2x_n} = \frac{\left(x_n + \frac{a}{x_n}\right)}{2}$

(b) $x_1 = 2$, $x_2 = 1.75$, $x_3 = 1.732142857$, $x_4 = 1.73205081$, $x_5 = 1.732050808$; we are finding the positive number where $x^2 - 3 = 0$; that is, where $x^2 = 3$, $x > 0$, or where $x = \sqrt{3}$.

134. $x_1 = 1$, $x_2 = 1 + \cos(1) = 1.540302306$, $x_3 = 1.540302306 + \cos(1 + \cos(1)) = 1.570791601$, $x_4 = 1.570791601 + \cos(1.570791601) = 1.570796327 = \frac{\pi}{2}$ to 9 decimal places. After a few steps, the arc (x_{n-1}) and line segment $\cos(x_{n-1})$ are nearly the same as the quarter circle.

135-146. Example CAS Commands:

Mathematica: (sequence functions may vary):

Clear[a, n]
a[n_]; = $n^{1/n}$
first25= Table[N[a[n]],{n, 1, 25}]
Limit[a[n], n → 8]

Mathematica: (sequence functions may vary):

Clear[a, n]
a[n_]; = $n^{1/n}$
first25= Table[N[a[n]],{n, 1, 25}]
Limit[a[n], n → 8]

The last command (Limit) will not always work in Mathematica. You could also explore the limit by enlarging your table to more than the first 25 values.

If you know the limit (1 in the above example), to determine how far to go to have all further terms within 0.01 of the limit, do the following.

Clear[minN, lim]
lim= 1
Do[{diff=Abs[a[n] − lim], If[diff < .01, {minN= n, Abort[]}]}, {n, 2, 1000}]
minN

For sequences that are given recursively, the following code is suggested. The portion of the command a[n_]:=a[n] stores the elements of the sequence and helps to streamline computation.

Clear[a, n]
a[1]= 1;
a[n_]; = a[n]= a[n − 1] + $(1/5)^{(n-1)}$
first25= Table[N[a[n]], {n, 1, 25}]

The limit command does not work in this case, but the limit can be observed as 1.25.

Clear[minN, lim]
lim= 1.25
Do[{diff=Abs[a[n] − lim], If[diff < .01, {minN= n, Abort[]}]}, {n, 2, 1000}]
minN

10.2 INFINITE SERIES

1. $s_n = \frac{a(1-r^n)}{(1-r)} = \frac{2\left(1-\left(\frac{1}{3}\right)^n\right)}{1-\left(\frac{1}{3}\right)} \Rightarrow \lim_{n\to\infty} s_n = \frac{2}{1-\left(\frac{1}{3}\right)} = 3$

2. $s_n = \frac{a(1-r^n)}{(1-r)} = \frac{\left(\frac{9}{100}\right)\left(1-\left(\frac{1}{100}\right)^n\right)}{1-\left(\frac{1}{100}\right)} \Rightarrow \lim_{n\to\infty} s_n = \frac{\left(\frac{9}{100}\right)}{1-\left(\frac{1}{100}\right)} = \frac{1}{11}$

3. $s_n = \frac{a(1-r^n)}{(1-r)} = \frac{1-\left(-\frac{1}{2}\right)^n}{1-\left(-\frac{1}{2}\right)} \Rightarrow \lim_{n\to\infty} s_n = \frac{1}{\left(\frac{3}{2}\right)} = \frac{2}{3}$

4. $s_n = \frac{1-(-2)^n}{1-(-2)}$, a geometric series where $|r| > 1 \Rightarrow$ divergence

5. $\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2} \Rightarrow s_n = \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \ldots + \left(\frac{1}{n+1}-\frac{1}{n+2}\right) = \frac{1}{2} - \frac{1}{n+2} \Rightarrow \lim_{n\to\infty} s_n = \frac{1}{2}$

6. $\frac{5}{n(n+1)} = \frac{5}{n} - \frac{5}{n+1} \Rightarrow s_n = \left(5 - \frac{5}{2}\right) + \left(\frac{5}{2} - \frac{5}{3}\right) + \left(\frac{5}{3} - \frac{5}{4}\right) + \ldots + \left(\frac{5}{n-1} - \frac{5}{n}\right) + \left(\frac{5}{n} - \frac{5}{n+1}\right) = 5 - \frac{5}{n+1}$
$\Rightarrow \lim_{n \to \infty} s_n = 5$

7. $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \ldots$, the sum of this geometric series is $\frac{1}{1 - \left(-\frac{1}{4}\right)} = \frac{1}{1 + \left(\frac{1}{4}\right)} = \frac{4}{5}$

8. $\frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots$, the sum of this geometric series is $\frac{\left(\frac{1}{16}\right)}{1 - \left(\frac{1}{4}\right)} = \frac{1}{12}$

9. $\frac{7}{4} + \frac{7}{16} + \frac{7}{64} + \ldots$, the sum of this geometric series is $\frac{\left(\frac{7}{4}\right)}{1 - \left(\frac{1}{4}\right)} = \frac{7}{3}$

10. $5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \ldots$, the sum of this geometric series is $\frac{5}{1 - \left(-\frac{1}{4}\right)} = 4$

11. $(5+1) + \left(\frac{5}{2} + \frac{1}{3}\right) + \left(\frac{5}{4} + \frac{1}{9}\right) + \left(\frac{5}{8} + \frac{1}{27}\right) + \ldots$, is the sum of two geometric series; the sum is
$\frac{5}{1 - \left(\frac{1}{2}\right)} + \frac{1}{1 - \left(\frac{1}{3}\right)} = 10 + \frac{3}{2} = \frac{23}{2}$

12. $(5-1) + \left(\frac{5}{2} - \frac{1}{3}\right) + \left(\frac{5}{4} - \frac{1}{9}\right) + \left(\frac{5}{8} - \frac{1}{27}\right) + \ldots$, is the difference of two geometric series; the sum is
$\frac{5}{1 - \left(\frac{1}{2}\right)} - \frac{1}{1 - \left(\frac{1}{3}\right)} = 10 - \frac{3}{2} = \frac{17}{2}$

13. $(1+1) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{4} + \frac{1}{25}\right) + \left(\frac{1}{8} - \frac{1}{125}\right) + \ldots$, is the sum of two geometric series; the sum is
$\frac{1}{1 - \left(\frac{1}{2}\right)} + \frac{1}{1 + \left(\frac{1}{5}\right)} = 2 + \frac{5}{6} = \frac{17}{6}$

14. $2 + \frac{4}{5} + \frac{8}{25} + \frac{16}{125} + \ldots = 2\left(1 + \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \ldots\right)$; the sum of this geometric series is $2\left(\frac{1}{1 - \left(\frac{2}{5}\right)}\right) = \frac{10}{3}$

15. Series is geometric with $r = \frac{2}{5} \Rightarrow \left|\frac{2}{5}\right| < 1 \Rightarrow$ Converges to $\frac{1}{1 - \frac{2}{5}} = \frac{5}{3}$

16. Series is geometric with $r = -3 \Rightarrow \left|-3\right| > 1 \Rightarrow$ Diverges

17. Series is geometric with $r = \frac{1}{8} \Rightarrow \left|\frac{1}{8}\right| < 1 \Rightarrow$ Converges to $\frac{\frac{1}{8}}{1 - \frac{1}{8}} = \frac{1}{7}$

18. Series is geometric with $r = -\frac{2}{3} \Rightarrow \left|-\frac{2}{3}\right| < 1 \Rightarrow$ Converges to $\frac{-\frac{2}{3}}{1 - \left(-\frac{2}{3}\right)} = -\frac{2}{5}$

19. $0.\overline{23} = \sum_{n=0}^{\infty} \frac{23}{100}\left(\frac{1}{10^2}\right)^n = \frac{\left(\frac{23}{100}\right)}{1 - \left(\frac{1}{100}\right)} = \frac{23}{99}$

20. $0.\overline{234} = \sum_{n=0}^{\infty} \frac{234}{1000}\left(\frac{1}{10^3}\right)^n = \frac{\left(\frac{234}{1000}\right)}{1 - \left(\frac{1}{1000}\right)} = \frac{234}{999}$

21. $0.\overline{7} = \sum_{n=0}^{\infty} \frac{7}{10}\left(\frac{1}{10}\right)^n = \frac{\left(\frac{7}{10}\right)}{1 - \left(\frac{1}{10}\right)} = \frac{7}{9}$

22. $0.\overline{d} = \sum_{n=0}^{\infty} \frac{d}{10}\left(\frac{1}{10}\right)^n = \frac{\left(\frac{d}{10}\right)}{1 - \left(\frac{1}{10}\right)} = \frac{d}{9}$

23. $0.0\overline{6} = \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)\left(\frac{6}{10}\right)\left(\frac{1}{10}\right)^n = \frac{\left(\frac{6}{100}\right)}{1 - \left(\frac{1}{10}\right)} = \frac{6}{90} = \frac{1}{15}$

24. $1.\overline{414} = 1 + \sum_{n=0}^{\infty} \frac{414}{1000}\left(\frac{1}{10^3}\right)^n = 1 + \frac{\left(\frac{414}{1000}\right)}{1 - \left(\frac{1}{1000}\right)} = 1 + \frac{414}{999} = \frac{1413}{999}$

25. $1.24\overline{123} = \frac{124}{100} + \sum_{n=0}^{\infty} \frac{123}{10^5}\left(\frac{1}{10^3}\right)^n = \frac{124}{100} + \frac{\left(\frac{123}{10^5}\right)}{1-\left(\frac{1}{10^3}\right)} = \frac{124}{100} + \frac{123}{10^5 - 10^2} = \frac{124}{100} + \frac{123}{99{,}900} = \frac{123{,}999}{99{,}900} = \frac{41{,}333}{33{,}300}$

26. $3.\overline{142857} = 3 + \sum_{n=0}^{\infty} \frac{142{,}857}{10^6}\left(\frac{1}{10^6}\right)^n = 3 + \frac{\left(\frac{142{,}857}{10^6}\right)}{1-\left(\frac{1}{10^6}\right)} = 3 + \frac{142{,}857}{10^6 - 1} = \frac{3{,}142{,}854}{999{,}999} = \frac{116{,}402}{37{,}037}$

27. $\lim_{n\to\infty} \frac{n}{n+10} = \lim_{n\to\infty} \frac{1}{1} = 1 \neq 0 \Rightarrow$ diverges

28. $\lim_{n\to\infty} \frac{n(n+1)}{(n+2)(n+3)} = \lim_{n\to\infty} \frac{n^2+n}{n^2+5n+6} = \lim_{n\to\infty} \frac{2n+1}{2n+5} = \lim_{n\to\infty} \frac{2}{2} = 1 \neq 0 \Rightarrow$ diverges

29. $\lim_{n\to\infty} \frac{1}{n+4} = 0 \Rightarrow$ test inconclusive

30. $\lim_{n\to\infty} \frac{n}{n^2+3} = \lim_{n\to\infty} \frac{1}{2n} = 0 \Rightarrow$ test inconclusive

31. $\lim_{n\to\infty} \cos\frac{1}{n} = \cos 0 = 1 \neq 0 \Rightarrow$ diverges

32. $\lim_{n\to\infty} \frac{e^n}{e^n+n} = \lim_{n\to\infty} \frac{e^n}{e^n+1} = \lim_{n\to\infty} \frac{e^n}{e^n} = \lim_{n\to\infty} \frac{1}{1} = 1 \neq 0 \Rightarrow$ diverges

33. $\lim_{n\to\infty} \ln\frac{1}{n} = -\infty \neq 0 \Rightarrow$ diverges

34. $\lim_{n\to\infty} \cos n\pi =$ does not exist $\Rightarrow$ diverges

35. $s_k = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \ldots + \left(\frac{1}{k-1}-\frac{1}{k}\right) + \left(\frac{1}{k}-\frac{1}{k+1}\right) = 1 - \frac{1}{k+1} \Rightarrow \lim_{k\to\infty} s_k$
$= \lim_{k\to\infty} \left(1-\frac{1}{k+1}\right) = 1$, series converges to 1

36. $s_k = \left(\frac{3}{1}-\frac{3}{4}\right) + \left(\frac{3}{4}-\frac{3}{9}\right) + \left(\frac{3}{9}-\frac{3}{16}\right) + \ldots + \left(\frac{3}{(k-1)^2}-\frac{3}{k^2}\right) + \left(\frac{3}{k^2}-\frac{3}{(k+1)^2}\right) = 3 - \frac{3}{(k+1)^2} \Rightarrow \lim_{k\to\infty} s_k$
$= \lim_{k\to\infty} \left(3-\frac{3}{(k+1)^2}\right) = 3$, series converges to 3

37. $s_k = \left(\ln\sqrt{2}-\ln\sqrt{1}\right) + \left(\ln\sqrt{3}-\ln\sqrt{2}\right) + \left(\ln\sqrt{4}-\ln\sqrt{3}\right) + \ldots + \left(\ln\sqrt{k}-\ln\sqrt{k-1}\right) + \left(\ln\sqrt{k+1}-\ln\sqrt{k}\right)$
$= \ln\sqrt{k+1} - \ln\sqrt{1} = \ln\sqrt{k+1} \Rightarrow \lim_{k\to\infty} s_k = \lim_{k\to\infty} \ln\sqrt{k+1} = \infty$; series diverges

38. $s_k = (\tan 1 - \tan 0) + (\tan 2 - \tan 1) + (\tan 3 - \tan 2) + \ldots + (\tan k - \tan(k-1)) + (\tan(k+1) - \tan k)$
$= \tan(k+1) - \tan 0 = \tan(k+1) \Rightarrow \lim_{k\to\infty} sk = \lim_{k\to\infty} \tan(k+1) =$ does not exist; series diverges

39. $s_k = \left(\cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{1}{3}\right)\right) + \left(\cos^{-1}\left(\frac{1}{3}\right) - \cos^{-1}\left(\frac{1}{4}\right)\right) + \left(\cos^{-1}\left(\frac{1}{4}\right) - \cos^{-1}\left(\frac{1}{5}\right)\right) + \ldots$
$+ \left(\cos^{-1}\left(\frac{1}{k}\right) - \cos^{-1}\left(\frac{1}{k+1}\right)\right) + \left(\cos^{-1}\left(\frac{1}{k+1}\right) - \cos^{-1}\left(\frac{1}{k+2}\right)\right) = \frac{\pi}{3} - \cos^{-1}\left(\frac{1}{k+2}\right)$
$\Rightarrow \lim_{k\to\infty} s_k = \lim_{k\to\infty} \left[\frac{\pi}{3} - \cos^{-1}\left(\frac{1}{k+2}\right)\right] = \frac{\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$, series converges to $\frac{\pi}{6}$

40. $s_k = \left(\sqrt{5}-\sqrt{4}\right) + \left(\sqrt{6}-\sqrt{5}\right) + \left(\sqrt{7}-\sqrt{6}\right) + \ldots + \left(\sqrt{k+3}-\sqrt{k+2}\right) + \left(\sqrt{k+4}-\sqrt{k+3}\right)$
$= \sqrt{k+4} - 2 \Rightarrow \lim_{k\to\infty} s_k = \lim_{k\to\infty} \left[\sqrt{k+4} - 2\right] = \infty$; series diverges

41. $\frac{4}{(4n-3)(4n+1)} = \frac{1}{4n-3} - \frac{1}{4n+1} \Rightarrow s_k = \left(1 - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{13}\right) + \ldots + \left(\frac{1}{4k-7} - \frac{1}{4k-3}\right)$
$+ \left(\frac{1}{4k-3} - \frac{1}{4k+1}\right) = 1 - \frac{1}{4k+1} \Rightarrow \lim_{k \to \infty} s_k = \lim_{k \to \infty} \left(1 - \frac{1}{4k+1}\right) = 1$

42. $\frac{6}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1} = \frac{A(2n+1) + B(2n-1)}{(2n-1)(2n+1)} \Rightarrow A(2n+1) + B(2n-1) = 6 \Rightarrow (2A + 2B)n + (A - B) = 6$
$\Rightarrow \begin{cases} 2A + 2B = 0 \\ A - B = 6 \end{cases} \Rightarrow \begin{cases} A + B = 0 \\ A - B = 6 \end{cases} \Rightarrow 2A = 6 \Rightarrow A = 3$ and $B = -3$. Hence, $\sum_{n=1}^{k} \frac{6}{(2n-1)(2n+1)} = 3 \sum_{n=1}^{k} \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$
$= 3\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \ldots - \frac{1}{2(k-1)+1} + \frac{1}{2k-1} - \frac{1}{2k+1}\right) = 3\left(1 - \frac{1}{2k+1}\right) \Rightarrow$ the sum is
$\lim_{k \to \infty} 3\left(1 - \frac{1}{2k+1}\right) = 3$

43. $\frac{40n}{(2n-1)^2(2n+1)^2} = \frac{A}{(2n-1)} + \frac{B}{(2n-1)^2} + \frac{C}{(2n+1)} + \frac{D}{(2n+1)^2} = \frac{A(2n-1)(2n+1)^2 + B(2n+1)^2 + C(2n+1)(2n-1)^2 + D(2n-1)^2}{(2n-1)^2(2n+1)^2}$
$\Rightarrow A(2n-1)(2n+1)^2 + B(2n+1)^2 + C(2n+1)(2n-1)^2 + D(2n-1)^2 = 40n$
$\Rightarrow A\left(8n^3 + 4n^2 - 2n - 1\right) + B\left(4n^2 + 4n + 1\right) + C\left(8n^3 - 4n^2 - 2n + 1\right) = D\left(4n^2 - 4n + 1\right) = 40n$
$\Rightarrow (8A + 8C)n^3 + (4A + 4B - 4C + 4D)n^2 + (-2A + 4B - 2C - 4D)n + (-A + B + C + D) = 40n$
$\Rightarrow \begin{cases} 8A + 8C = 0 \\ 4A + 4B - 4C + 4D = 0 \\ -2A + 4B - 2C - 4D = 40 \\ -A + B + C + D = 0 \end{cases} \Rightarrow \begin{cases} 8A + 8C = 0 \\ A + B - C + D = 0 \\ -A + 2B - C - 2D = 20 \\ -A + B + C + D = 0 \end{cases} \Rightarrow \begin{cases} B + D = 0 \\ 2B - 2D = 20 \end{cases} \Rightarrow 4B = 20 \Rightarrow B = 5$
and $D = -5 \Rightarrow \begin{cases} A + C = 0 \\ -A + 5 + C - 5 = 0 \end{cases} \Rightarrow C = 0$ and $A = 0$. Hence, $\sum_{n=1}^{k} \left[\frac{40n}{(2n-1)^2(2n+1)^2}\right]$
$= 5 \sum_{n=1}^{k} \left[\frac{1}{(2n-1)^2} - \frac{1}{(2n+1)^2}\right] = 5\left(\frac{1}{1} - \frac{1}{9} + \frac{1}{9} - \frac{1}{25} + \frac{1}{25} - \ldots - \frac{1}{(2(k-1)+1)^2} + \frac{1}{(2k-1)^2} - \frac{1}{(2k+1)^2}\right)$
$= 5\left(1 - \frac{1}{(2k+1)^2}\right) \Rightarrow$ the sum is $\lim_{n \to \infty} 5\left(1 - \frac{1}{(2k+1)^2}\right) = 5$

44. $\frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2} \Rightarrow s_k = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{16}\right) + \ldots + \left[\frac{1}{(k-1)^2} - \frac{1}{k^2}\right] + \left[\frac{1}{k^2} - \frac{1}{(k+1)^2}\right]$
$\Rightarrow \lim_{k \to \infty} s_k = \lim_{k \to \infty} \left[1 - \frac{1}{(k+1)^2}\right] = 1$

45. $s_k = \left(1 - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) + \ldots + \left(\frac{1}{\sqrt{k-1}} + \frac{1}{\sqrt{k}}\right) + \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) = 1 - \frac{1}{\sqrt{k+1}}$
$\Rightarrow \lim_{k \to \infty} s_k = \lim_{k \to \infty} \left(1 - \frac{1}{\sqrt{k+1}}\right) = 1$

46. $s_k = \left(\frac{1}{2} - \frac{1}{2^{1/2}}\right) + \left(\frac{1}{2^{1/2}} - \frac{1}{2^{1/3}}\right) + \left(\frac{1}{2^{1/3}} - \frac{1}{2^{1/4}}\right) + \ldots + \left(\frac{1}{2^{1/(k-1)}} - \frac{1}{2^{1/k}}\right) + \left(\frac{1}{2^{1/k}} - \frac{1}{2^{1/(k+1)}}\right) = \frac{1}{2} - \frac{1}{2^{1/(k+1)}}$
$\Rightarrow \lim_{k \to \infty} s_k = \frac{1}{2} - \frac{1}{1} = -\frac{1}{2}$

47. $s_k = \left(\frac{1}{\ln 3} - \frac{1}{\ln 2}\right) + \left(\frac{1}{\ln 4} - \frac{1}{\ln 3}\right) + \left(\frac{1}{\ln 5} - \frac{1}{\ln 4}\right) + \ldots + \left(\frac{1}{\ln(k+1)} - \frac{1}{\ln k}\right) + \left(\frac{1}{\ln(k+2)} - \frac{1}{\ln(k+1)}\right)$
$= -\frac{1}{\ln 2} + \frac{1}{\ln(k+2)} \Rightarrow \lim_{k \to \infty} s_k = -\frac{1}{\ln 2}$

48. $s_k = [\tan^{-1}(1) - \tan^{-1}(2)] + [\tan^{-1}(2) - \tan^{-1}(3)] + \ldots + [\tan^{-1}(k-1) - \tan^{-1}(k)]$
$+ [\tan^{-1}(k) - \tan^{-1}(k+1)] = \tan^{-1}(1) - \tan^{-1}(k+1) \Rightarrow \lim_{k \to \infty} s_k = \tan^{-1}(1) - \frac{\pi}{2} = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$

49. convergent geometric series with sum $\frac{1}{1 - \left(\frac{1}{\sqrt{2}}\right)} = \frac{\sqrt{2}}{\sqrt{2}-1} = 2 + \sqrt{2}$

50. divergent geometric series with $|r| = \sqrt{2} > 1$

51. convergent geometric series with sum $\frac{\left(\frac{3}{2}\right)}{1 - \left(-\frac{1}{2}\right)} = 1$

52. $\lim\limits_{n \to \infty} (-1)^{n+1} n \neq 0 \ \Rightarrow$ diverges

53. $\lim\limits_{n \to \infty} \cos(n\pi) = \lim\limits_{n \to \infty} (-1)^n \neq 0 \ \Rightarrow$ diverges

54. $\cos(n\pi) = (-1)^n \ \Rightarrow$ convergent geometric series with sum $\frac{1}{1-\left(-\frac{1}{5}\right)} = \frac{5}{6}$

55. convergent geometric series with sum $\frac{1}{1-\left(\frac{1}{e^2}\right)} = \frac{e^2}{e^2-1}$

56. $\lim\limits_{n \to \infty} \ln \frac{1}{3^n} = -\infty \neq 0 \ \Rightarrow$ diverges

57. convergent geometric series with sum $\frac{2}{1-\left(\frac{1}{10}\right)} - 2 = \frac{20}{9} - \frac{18}{9} = \frac{2}{9}$

58. convergent geometric series with sum $\frac{1}{1-\left(\frac{1}{x}\right)} = \frac{x}{x-1}$

59. difference of two geometric series with sum $\frac{1}{1-\left(\frac{2}{3}\right)} - \frac{1}{1-\left(\frac{1}{3}\right)} = 3 - \frac{3}{2} = \frac{3}{2}$

60. $\lim\limits_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = \lim\limits_{n \to \infty} \left(1 + \frac{-1}{n}\right)^n = e^{-1} \neq 0 \ \Rightarrow$ diverges

61. $\lim\limits_{n \to \infty} \frac{n!}{1000^n} = \infty \neq 0 \ \Rightarrow$ diverges

62. $\lim\limits_{n \to \infty} \frac{n^n}{n!} = \lim\limits_{n \to \infty} \frac{n \cdot n \cdots n}{1 \cdot 2 \cdots n} > \lim\limits_{n \to \infty} n = \infty \ \Rightarrow$ diverges

63. $\sum\limits_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = \sum\limits_{n=1}^{\infty} \frac{2^n}{4^n} + \sum\limits_{n=1}^{\infty} \frac{3^n}{4^n} = \sum\limits_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum\limits_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$; both $= \sum\limits_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ and $\sum\limits_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$ are geometric series, and both converge since $r = \frac{1}{2} \Rightarrow \left|\frac{1}{2}\right| < 1$ and $r = \frac{3}{4} \Rightarrow \left|\frac{3}{4}\right| < 1$, respectivley $\Rightarrow \sum\limits_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ and $\sum\limits_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = \frac{\frac{3}{4}}{1-\frac{3}{4}} = 3 \Rightarrow$ $\sum\limits_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = 1 + 3 = 4$ by Theorem 8, part (1)

64. $\lim\limits_{n\to\infty} \frac{2^n + 4^n}{3^n + 4^n} = \lim\limits_{n\to\infty} \frac{\frac{2^n}{4^n} + 1}{\frac{3^n}{4^n} + 1} = \lim\limits_{n\to\infty} \frac{\left(\frac{1}{2}\right)^n + 1}{\left(\frac{3}{4}\right)^n + 1} = \frac{1}{1} = 1 \neq 0 \Rightarrow$ diverges by n^{th} term test for divergence

65. $\sum\limits_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum\limits_{n=1}^{\infty} [\ln(n) - \ln(n+1)] \ \Rightarrow \ s_k = [\ln(1) - \ln(2)] + [\ln(2) - \ln(3)] + [\ln(3) - \ln(4)] + \ldots$ $+ [\ln(k-1) - \ln(k)] + [\ln(k) - \ln(k+1)] = -\ln(k+1) \ \Rightarrow \ \lim\limits_{k \to \infty} s_k = -\infty, \ \Rightarrow$ diverges

66. $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \ln\left(\frac{n}{2n+1}\right) = \ln\left(\frac{1}{2}\right) \neq 0 \ \Rightarrow$ diverges

67. convergent geometric series with sum $\frac{1}{1-\left(\frac{e}{\pi}\right)} = \frac{\pi}{\pi - e}$

68. divergent geometric series with $|r| = \frac{e^\pi}{\pi^e} \approx \frac{23.141}{22.459} > 1$

69. $\sum\limits_{n=0}^{\infty} (-1)^n x^n = \sum\limits_{n=0}^{\infty} (-x)^n$; $a = 1$, $r = -x$; converges to $\frac{1}{1-(-x)} = \frac{1}{1+x}$ for $|x| < 1$

70. $\sum\limits_{n=0}^{\infty} (-1)^n x^{2n} = \sum\limits_{n=0}^{\infty} \left(-x^2\right)^n$; $a = 1$, $r = -x^2$; converges to $\frac{1}{1+x^2}$ for $|x| < 1$

71. $a = 3$, $r = \frac{x-1}{2}$; converges to $\frac{3}{1-\left(\frac{x-1}{2}\right)} = \frac{6}{3-x}$ for $-1 < \frac{x-1}{2} < 1$ or $-1 < x < 3$

72. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2}\left(\frac{1}{3+\sin x}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{-1}{3+\sin x}\right)^n$; $a = \frac{1}{2}$, $r = \frac{-1}{3+\sin x}$; converges to $\frac{\left(\frac{1}{2}\right)}{1-\left(\frac{-1}{3+\sin x}\right)}$
$= \frac{3+\sin x}{2(4+\sin x)} = \frac{3+\sin x}{8+2\sin x}$ for all x $\left(\text{since } \frac{1}{4} \le \frac{1}{3+\sin x} \le \frac{1}{2} \text{ for all } x\right)$

73. $a = 1$, $r = 2x$; converges to $\frac{1}{1-2x}$ for $|2x| < 1$ or $|x| < \frac{1}{2}$

74. $a = 1$, $r = -\frac{1}{x^2}$; converges to $\frac{1}{1-\left(\frac{-1}{x^2}\right)} = \frac{x^2}{x^2+1}$ for $\left|\frac{1}{x^2}\right| < 1$ or $|x| > 1$.

75. $a = 1$, $r = -(x+1)^n$; converges to $\frac{1}{1+(x+1)} = \frac{1}{2+x}$ for $|x+1| < 1$ or $-2 < x < 0$

76. $a = 1$, $r = \frac{3-x}{2}$; converges to $\frac{1}{1-\left(\frac{3-x}{2}\right)} = \frac{2}{x-1}$ for $\left|\frac{3-x}{2}\right| < 1$ or $1 < x < 5$

77. $a = 1$, $r = \sin x$; converges to $\frac{1}{1-\sin x}$ for $x \ne (2k+1)\frac{\pi}{2}$, k an integer

78. $a = 1$, $r = \ln x$; converges to $\frac{1}{1-\ln x}$ for $|\ln x| < 1$ or $e^{-1} < x < e$

79. (a) $\sum_{n=-2}^{\infty} \frac{1}{(n+4)(n+5)}$ (b) $\sum_{n=0}^{\infty} \frac{1}{(n+2)(n+3)}$ (c) $\sum_{n=5}^{\infty} \frac{1}{(n-3)(n-2)}$

80. (a) $\sum_{n=-1}^{\infty} \frac{5}{(n+2)(n+3)}$ (b) $\sum_{n=3}^{\infty} \frac{5}{(n-2)(n-1)}$ (c) $\sum_{n=20}^{\infty} \frac{5}{(n-19)(n-18)}$

81. (a) one example is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)} = 1$

(b) one example is $-\frac{3}{2} - \frac{3}{4} - \frac{3}{8} - \frac{3}{16} - \ldots = \frac{\left(-\frac{3}{2}\right)}{1-\left(\frac{1}{2}\right)} = -3$

(c) one example is $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \ldots = 1 - \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)} = 0.$

82. The series $\sum_{n=0}^{\infty} k\left(\frac{1}{2}\right)^{n+1}$ is a geometric series whose sum is $\frac{\left(\frac{k}{2}\right)}{1-\left(\frac{1}{2}\right)} = k$ where k can be any positive or negative number.

83. Let $a_n = b_n = \left(\frac{1}{2}\right)^n$. Then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$, while $\sum_{n=1}^{\infty} \left(\frac{a_n}{b_n}\right) = \sum_{n=1}^{\infty} (1)$ diverges.

84. Let $a_n = b_n = \left(\frac{1}{2}\right)^n$. Then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$, while $\sum_{n=1}^{\infty} (a_n b_n) = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{3} \ne AB$.

85. Let $a_n = \left(\frac{1}{4}\right)^n$ and $b_n = \left(\frac{1}{2}\right)^n$. Then $A = \sum_{n=1}^{\infty} a_n = \frac{1}{3}$, $B = \sum_{n=1}^{\infty} b_n = 1$ and $\sum_{n=1}^{\infty} \left(\frac{a_n}{b_n}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1 \ne \frac{A}{B}$.

86. Yes: $\sum \left(\frac{1}{a_n}\right)$ diverges. The reasoning: $\sum a_n$ converges $\Rightarrow a_n \to 0 \Rightarrow \frac{1}{a_n} \to \infty \Rightarrow \sum \left(\frac{1}{a_n}\right)$ diverges by the nth-Term Test.

87. Since the sum of a finite number of terms is finite, adding or subtracting a finite number of terms from a series that diverges does not change the divergence of the series.

88. Let $A_n = a_1 + a_2 + \ldots + a_n$ and $\lim\limits_{n\to\infty} A_n = A$. Assume $\sum (a_n + b_n)$ converges to S. Let $S_n = (a_1 + b_1) + (a_2 + b_2) + \ldots + (a_n + b_n) \Rightarrow S_n = (a_1 + a_2 + \ldots + a_n) + (b_1 + b_2 + \ldots + b_n)$ $\Rightarrow b_1 + b_2 + \ldots + b_n = S_n - A_n \Rightarrow \lim\limits_{n\to\infty} (b_1 + b_2 + \ldots + b_n) = S - A \Rightarrow \sum b_n$ converges. This contradicts the assumption that $\sum b_n$ diverges; therefore, $\sum (a_n + b_n)$ diverges.

89. (a) $\frac{2}{1-r} = 5 \Rightarrow \frac{2}{5} = 1 - r \Rightarrow r = \frac{3}{5}$; $2 + 2\left(\frac{3}{5}\right) + 2\left(\frac{3}{5}\right)^2 + \ldots$

(b) $\frac{\left(\frac{13}{2}\right)}{1-r} = 5 \Rightarrow \frac{13}{10} = 1 - r \Rightarrow r = -\frac{3}{10}$; $\frac{13}{2} - \frac{13}{2}\left(\frac{3}{10}\right) + \frac{13}{2}\left(\frac{3}{10}\right)^2 - \frac{13}{2}\left(\frac{3}{10}\right)^3 + \ldots$

90. $1 + e^b + e^{2b} + \ldots = \frac{1}{1-e^b} = 9 \Rightarrow \frac{1}{9} = 1 - e^b \Rightarrow e^b = \frac{8}{9} \Rightarrow b = \ln\left(\frac{8}{9}\right)$

91. $s_n = 1 + 2r + r^2 + 2r^3 + r^4 + 2r^5 + \ldots + r^{2n} + 2r^{2n+1}$, n = 0, 1, …
$\Rightarrow s_n = (1 + r^2 + r^4 + \ldots + r^{2n}) + (2r + 2r^3 + 2r^5 + \ldots + 2r^{2n+1}) \Rightarrow \lim\limits_{n\to\infty} s_n = \frac{1}{1-r^2} + \frac{2r}{1-r^2}$
$= \frac{1+2r}{1-r^2}$, if $|r^2| < 1$ or $|r| < 1$

92. $L - s_n = \frac{a}{1-r} - \frac{a(1-r^n)}{1-r} = \frac{ar^n}{1-r}$

93. area $= 2^2 + \left(\sqrt{2}\right)^2 + (1)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \ldots = 4 + 2 + 1 + \frac{1}{2} + \ldots = \frac{4}{1-\frac{1}{2}} = 8\ \text{m}^2$

94. (a) $L_1 = 3, L_2 = 3\left(\frac{4}{3}\right), L_3 = 3\left(\frac{4}{3}\right)^2, \ldots, L_n = 3\left(\frac{4}{3}\right)^{n-1} \Rightarrow \lim\limits_{n\to\infty} L_n = \lim\limits_{n\to\infty} 3\left(\frac{4}{3}\right)^{n-1} = \infty$

(b) Using the fact that the area of an equilateral triangle of side length s is $\frac{\sqrt{3}}{4}s^2$, we see that $A_1 = \frac{\sqrt{3}}{4}$,
$A_2 = A_1 + 3\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3}\right)^2 = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}$, $A_3 = A_2 + 3(4)\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3^2}\right)^2 = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} + \frac{\sqrt{3}}{27}$,
$A_4 = A_3 + 3(4)^2\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3^3}\right)^2$, $A_5 = A_4 + 3(4)^3\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3^4}\right)^2, \ldots,$
$A_n = \frac{\sqrt{3}}{4} + \sum\limits_{k=2}^{n} 3(4)^{k-2}\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3^2}\right)^{k-1} = \frac{\sqrt{3}}{4} + \sum\limits_{k=2}^{n} 3\sqrt{3}(4)^{k-3}\left(\frac{1}{9}\right)^{k-1} = \frac{\sqrt{3}}{4} + 3\sqrt{3}\left(\sum\limits_{k=2}^{n} \frac{4^{k-3}}{9^{k-1}}\right)$.
$\lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty}\left(\frac{\sqrt{3}}{4} + 3\sqrt{3}\left(\sum\limits_{k=2}^{n} \frac{4^{k-3}}{9^{k-1}}\right)\right) = \frac{\sqrt{3}}{4} + 3\sqrt{3}\left(\frac{\frac{1}{36}}{1-\frac{4}{9}}\right) = \frac{\sqrt{3}}{4} + 3\sqrt{3}\left(\frac{1}{20}\right) = \frac{\sqrt{3}}{4}\left(1 + \frac{3}{5}\right)$
$= \frac{\sqrt{3}}{4}\left(\frac{8}{5}\right) = \frac{8}{5}A_1$

10.3 THE INTEGRAL TEST

1. $f(x) = \frac{1}{x^2}$ is positive, continuous, and decreasing for $x \ge 1$; $\int_1^\infty \frac{1}{x^2}\,dx = \lim\limits_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim\limits_{b\to\infty} \left[-\frac{1}{x}\right]_1^b$
$= \lim\limits_{b\to\infty} \left(-\frac{1}{b} + 1\right) = 1 \Rightarrow \int_1^\infty \frac{1}{x^2}\,dx$ converges $\Rightarrow \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges

2. $f(x) = \frac{1}{x^{0.2}}$ is positive, continuous, and decreasing for $x \ge 1$; $\int_1^\infty \frac{1}{x^{0.2}}\,dx = \lim\limits_{b\to\infty} \int_1^b \frac{1}{x^{0.2}}\,dx = \lim\limits_{b\to\infty} \left[\frac{5}{4}x^{0.8}\right]_1^b$
$= \lim\limits_{b\to\infty} \left(\frac{5}{4}b^{0.8} - \frac{5}{4}\right) = \infty \Rightarrow \int_1^\infty \frac{1}{x^{0.2}}\,dx$ diverges $\Rightarrow \sum\limits_{n=1}^{\infty} \frac{1}{n^{0.2}}$ diverges

3. $f(x) = \frac{1}{x^2+4}$ is positive, continuous, and decreasing for $x \ge 1$; $\int_1^{\infty} \frac{1}{x^2+4}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2+4}\,dx = \lim_{b\to\infty} \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_1^b$
$= \lim_{b\to\infty} \left(\frac{1}{2}\tan^{-1}\frac{b}{2} - \frac{1}{2}\tan^{-1}\frac{1}{2}\right) = \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\frac{1}{2} \Rightarrow \int_1^{\infty} \frac{1}{x^2+4}\,dx$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+4}$ converges

4. $f(x) = \frac{1}{x+4}$ is positive, continuous, and decreasing for $x \ge 1$; $\int_1^{\infty} \frac{1}{x+4}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x+4}\,dx = \lim_{b\to\infty} \left[\ln|x+4|\right]_1^b$
$= \lim_{b\to\infty} (\ln|b+4| - \ln 5) = \infty \Rightarrow \int_1^{\infty} \frac{1}{x+4}\,dx$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n+4}$ diverges

5. $f(x) = e^{-2x}$ is positive, continuous, and decreasing for $x \ge 1$; $\int_1^{\infty} e^{-2x}\,dx = \lim_{b\to\infty} \int_1^b e^{-2x}\,dx = \lim_{b\to\infty} \left[-\frac{1}{2}e^{-2x}\right]_1^b$
$= \lim_{b\to\infty} \left(-\frac{1}{2e^{2b}} + \frac{1}{2e^2}\right) = \frac{1}{2e^2} \Rightarrow \int_1^{\infty} e^{-2x}\,dx$ converges $\Rightarrow \sum_{n=1}^{\infty} e^{-2n}$ converges

6. $f(x) = \frac{1}{x(\ln x)^2}$ is positive, continuous, and decreasing for $x \ge 2$; $\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{\ln x}\right]_2^b$
$= \lim_{b\to\infty} \left(-\frac{1}{\ln b} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2} \Rightarrow \int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$ converges $\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges

7. $f(x) = \frac{x}{x^2+4}$ is positive and continuous for $x \ge 1$, $f'(x) = \frac{4-x^2}{(x^2+4)^2} < 0$ for $x > 2$, thus f is decreasing for $x \ge 3$;
$\int_3^{\infty} \frac{x}{x^2+4}\,dx = \lim_{b\to\infty} \int_3^b \frac{x}{x^2+4}\,dx = \lim_{b\to\infty} \left[\frac{1}{2}\ln(x^2+4)\right]_3^b = \lim_{b\to\infty} \left(\frac{1}{2}\ln(b^2+4) - \frac{1}{2}\ln(13)\right) = \infty \Rightarrow \int_3^{\infty} \frac{x}{x^2+4}\,dx$
diverges $\Rightarrow \sum_{n=3}^{\infty} \frac{n}{n^2+4}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{n}{n^2+4} = \frac{1}{5} + \frac{2}{8} + \sum_{n=3}^{\infty} \frac{n}{n^2+4}$ diverges

8. $f(x) = \frac{\ln x^2}{x}$ is positive and continuous for $x \ge 2$, $f'(x) = \frac{2-\ln x^2}{x^2} < 0$ for $x > e$, thus f is decreasing for $x \ge 3$;
$\int_3^{\infty} \frac{\ln x^2}{x}\,dx = \lim_{b\to\infty} \int_3^b \frac{\ln x^2}{x}\,dx = \lim_{b\to\infty} \left[2(\ln x)\right]_3^b = \lim_{b\to\infty} (2(\ln b) - 2(\ln 3)) = \infty \Rightarrow \int_3^{\infty} \frac{\ln x^2}{x}\,dx$
diverges $\Rightarrow \sum_{n=3}^{\infty} \frac{\ln n^2}{n}$ diverges $\Rightarrow \sum_{n=2}^{\infty} \frac{\ln n^2}{n} = \frac{\ln 4}{2} + \sum_{n=3}^{\infty} \frac{\ln n^2}{n}$ diverges

9. $f(x) = \frac{x^2}{e^{x/3}}$ is positive and continuous for $x \ge 1$, $f'(x) = \frac{-x(x-6)}{3e^{x/3}} < 0$ for $x > 6$, thus f is decreasing for $x \ge 7$;
$\int_7^{\infty} \frac{x^2}{e^{x/3}}\,dx = \lim_{b\to\infty} \int_7^b \frac{x^2}{e^{x/3}}\,dx = \lim_{b\to\infty} \left[-\frac{3x^2}{e^{x/3}} - \frac{18x}{e^{x/3}} - \frac{54}{e^{x/3}}\right]_7^b = \lim_{b\to\infty} \left(\frac{-3b^2-18b-54}{e^{b/3}} + \frac{327}{e^{7/3}}\right) =$
$= \lim_{b\to\infty} \left(\frac{3(-6b-18)}{e^{b/3}}\right) + \frac{327}{e^{7/3}} = \lim_{b\to\infty} \left(\frac{-54}{e^{b/3}}\right) + \frac{327}{e^{7/3}} = \frac{327}{e^{7/3}} \Rightarrow \int_7^{\infty} \frac{x^2}{e^{x/3}}\,dx$ converges $\Rightarrow \sum_{n=7}^{\infty} \frac{n^2}{e^{n/3}}$ converges
$\Rightarrow \sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}} = \frac{1}{e^{1/3}} + \frac{4}{e^{2/3}} + \frac{9}{e^1} + \frac{16}{e^{4/3}} + \frac{25}{e^{5/3}} + \frac{36}{e^2} + \sum_{n=7}^{\infty} \frac{n^2}{e^{n/3}}$ converges

10. $f(x) = \frac{x-4}{x^2-2x+1} = \frac{x-4}{(x-1)^2}$ is continuous for $x \ge 2$, f is positive for $x > 4$, and $f'(x) = \frac{7-x}{(x-1)^3} < 0$ for $x > 7$, thus f is decreasing for $x \ge 8$; $\int_8^{\infty} \frac{x-4}{(x-1)^2}\,dx = \lim_{b\to\infty} \left[\int_8^b \frac{x-1}{(x-1)^2}\,dx - \int_8^b \frac{3}{(x-1)^2}\,dx\right] = \lim_{b\to\infty} \left[\int_8^b \frac{1}{x-1}\,dx - \int_8^b \frac{3}{(x-1)^2}\,dx\right]$
$= \lim_{b\to\infty} \left[\ln|x-1| + \frac{3}{x-1}\right]_8^b = \lim_{b\to\infty} \left(\ln|b-1| + \frac{3}{b-1} - \ln 7 - \frac{3}{7}\right) = \infty \Rightarrow \int_8^{\infty} \frac{x-4}{(x-1)^2}\,dx$ diverges
$\Rightarrow \sum_{n=8}^{\infty} \frac{n-4}{n^2-2n+1}$ diverges $\Rightarrow \sum_{n=2}^{\infty} \frac{n-4}{n^2-2n+1} = -2 - \frac{1}{4} + 0 + \frac{1}{16} + \frac{2}{25} + \frac{3}{36} + \sum_{n=8}^{\infty} \frac{n-4}{n^2-2n+1}$ diverges

11. converges; a geometric series with $r = \frac{1}{10} < 1$

12. converges; a geometric series with $r = \frac{1}{e} < 1$

13. diverges; by the nth-Term Test for Divergence, $\lim_{n\to\infty} \frac{n}{n+1} = 1 \ne 0$

14. diverges by the Integral Test; $\int_1^n \frac{5}{x+1}\,dx = 5\ln(n+1) - 5\ln 2 \Rightarrow \int_1^\infty \frac{5}{x+1}\,dx \to \infty$

15. diverges; $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} = 3\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series $\left(p = \frac{1}{2}\right)$

16. converges; $\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}} = -2\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which is a convergent p-series $\left(p = \frac{3}{2}\right)$

17. converges; a geometric series with $r = \frac{1}{8} < 1$

18. diverges; $\sum_{n=1}^{\infty} \frac{-8}{n} = -8\sum_{n=1}^{\infty} \frac{1}{n}$ and since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $-8\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

19. diverges by the Integral Test: $\int_2^n \frac{\ln x}{x}\,dx = \frac{1}{2}\left(\ln^2 n - \ln 2\right) \Rightarrow \int_2^\infty \frac{\ln x}{x}\,dx \to \infty$

20. diverges by the Integral Test: $\int_2^\infty \frac{\ln x}{\sqrt{x}}\,dx$; $\begin{bmatrix} t = \ln x \\ dt = \frac{dx}{x} \\ dx = e^t\,dt \end{bmatrix} \to \int_{\ln 2}^\infty te^{t/2}\,dt = \lim_{b\to\infty} \left[2te^{t/2} - 4e^{t/2}\right]_{\ln 2}^{b}$
$= \lim_{b\to\infty} \left[2e^{b/2}(b-2) - 2e^{(\ln 2)/2}(\ln 2 - 2)\right] = \infty$

21. converges; a geometric series with $r = \frac{2}{3} < 1$

22. diverges; $\lim_{n\to\infty} \frac{5^n}{4^n+3} = \lim_{n\to\infty} \frac{5^n \ln 5}{4^n \ln 4} = \lim_{n\to\infty} \left(\frac{\ln 5}{\ln 4}\right)\left(\frac{5}{4}\right)^n = \infty \neq 0$

23. diverges; $\sum_{n=0}^{\infty} \frac{-2}{n+1} = -2\sum_{n=0}^{\infty} \frac{1}{n+1}$, which diverges by the Integral Test

24. diverges by the Integral Test: $\int_1^n \frac{dx}{2x-1} = \frac{1}{2}\ln(2n-1) \to \infty$ as $n \to \infty$

25. diverges; $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n}{n+1} = \lim_{n\to\infty} \frac{2^n \ln 2}{1} = \infty \neq 0$

26. diverges by the Integral Test: $\int_1^n \frac{dx}{\sqrt{x}\left(\sqrt{x}+1\right)}$; $\begin{bmatrix} u = \sqrt{x}+1 \\ du = \frac{dx}{\sqrt{x}} \end{bmatrix} \to \int_2^{\sqrt{n}+1} \frac{du}{u} = \ln\left(\sqrt{n}+1\right) - \ln 2 \to \infty$ as $n \to \infty$

27. diverges; $\lim_{n\to\infty} \frac{\sqrt{n}}{\ln n} = \lim_{n\to\infty} \frac{\left(\frac{1}{2\sqrt{n}}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} = \infty \neq 0$

28. diverges; $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \neq 0$

29. diverges; a geometric series with $r = \frac{1}{\ln 2} \approx 1.44 > 1$

30. converges; a geometric series with $r = \frac{1}{\ln 3} \approx 0.91 < 1$

31. converges by the Integral Test: $\int_3^\infty \frac{\left(\frac{1}{x}\right)}{(\ln x)\sqrt{(\ln x)^2 - 1}}\,dx$; $\begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\,dx \end{bmatrix} \to \int_{\ln 3}^\infty \frac{1}{u\sqrt{u^2-1}}\,du$

$= \lim_{b\to\infty} [\sec^{-1}|u|]_{\ln 3}^{b} = \lim_{b\to\infty} [\sec^{-1} b - \sec^{-1}(\ln 3)] = \lim_{b\to\infty} \left[\cos^{-1}\left(\frac{1}{b}\right) - \sec^{-1}(\ln 3)\right]$
$= \cos^{-1}(0) - \sec^{-1}(\ln 3) = \frac{\pi}{2} - \sec^{-1}(\ln 3) \approx 1.1439$

32. converges by the Integral Test: $\int_1^\infty \frac{1}{x(1+\ln^2 x)}\,dx = \int_1^\infty \frac{\left(\frac{1}{x}\right)}{1+(\ln x)^2}\,dx;\ \begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\,dx \end{bmatrix} \to \int_0^\infty \frac{1}{1+u^2}\,du$
$= \lim_{b\to\infty} [\tan^{-1} u]_0^b = \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

33. diverges by the nth-Term Test for divergence; $\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{x\to 0} \frac{\sin x}{x} = 1 \neq 0$

34. diverges by the nth-Term Test for divergence; $\lim_{n\to\infty} n\tan\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{\left(-\frac{1}{n^2}\right)\sec^2\left(\frac{1}{n}\right)}{\left(-\frac{1}{n^2}\right)}$
$= \lim_{n\to\infty} \sec^2\left(\frac{1}{n}\right) = \sec^2 0 = 1 \neq 0$

35. converges by the Integral Test: $\int_1^\infty \frac{e^x}{1+e^{2x}}\,dx;\ \begin{bmatrix} u = e^x \\ du = e^x\,dx \end{bmatrix} \to \int_e^\infty \frac{1}{1+u^2}\,du = \lim_{n\to\infty} [\tan^{-1} u]_e^b$
$= \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} e) = \frac{\pi}{2} - \tan^{-1} e \approx 0.35$

36. converges by the Integral Test: $\int_1^\infty \frac{2}{1+e^x}\,dx;\ \begin{bmatrix} u = e^x \\ du = e^x\,dx \\ dx = \frac{1}{u}\,du \end{bmatrix} \to \int_e^\infty \frac{2}{u(1+u)}\,du = \int_e^\infty \left(\frac{2}{u} - \frac{2}{u+1}\right) du$
$= \lim_{b\to\infty} \left[2\ln\frac{u}{u+1}\right]_e^b = \lim_{b\to\infty} 2\ln\left(\frac{b}{b+1}\right) - 2\ln\left(\frac{e}{e+1}\right) = 2\ln 1 - 2\ln\left(\frac{e}{e+1}\right) = -2\ln\left(\frac{e}{e+1}\right) \approx 0.63$

37. converges by the Integral Test: $\int_1^\infty \frac{8\tan^{-1} x}{1+x^2}\,dx;\ \begin{bmatrix} u = \tan^{-1} x \\ du = \frac{dx}{1+x^2} \end{bmatrix} \to \int_{\pi/4}^{\pi/2} 8u\,du = [4u^2]_{\pi/4}^{\pi/2} = 4\left(\frac{\pi^2}{4} - \frac{\pi^2}{16}\right) = \frac{3\pi^2}{4}$

38. diverges by the Integral Test: $\int_1^\infty \frac{x}{x^2+1}\,dx;\ \begin{bmatrix} u = x^2+1 \\ du = 2x\,dx \end{bmatrix} \to \frac{1}{2}\int_2^\infty \frac{du}{4} = \lim_{b\to\infty} \left[\frac{1}{2}\ln u\right]_2^b = \lim_{b\to\infty} \frac{1}{2}(\ln b - \ln 2) = \infty$

39. converges by the Integral Test: $\int_1^\infty \operatorname{sech} x\,dx = 2\lim_{b\to\infty} \int_1^b \frac{e^x}{1+(e^x)^2}\,dx = 2\lim_{b\to\infty} [\tan^{-1} e^x]_1^b$
$= 2\lim_{b\to\infty} (\tan^{-1} e^b - \tan^{-1} e) = \pi - 2\tan^{-1} e \approx 0.71$

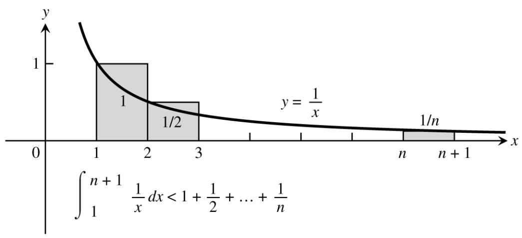
40. converges by the Integral Test: $\int_1^\infty \operatorname{sech}^2 x\,dx = \lim_{b\to\infty} \int_1^b \operatorname{sech}^2 x\,dx = \lim_{b\to\infty} [\tanh x]_1^b = \lim_{b\to\infty} (\tanh b - \tanh 1)$
$= 1 - \tanh 1 \approx 0.76$

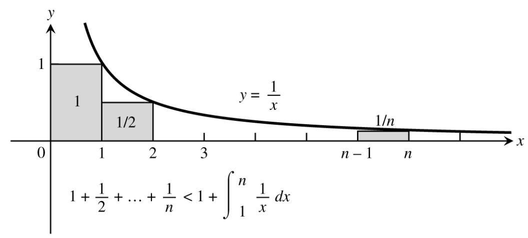
41. $\int_1^\infty \left(\frac{a}{x+2} - \frac{1}{x+4}\right) dx = \lim_{b\to\infty} [a\ln|x+2| - \ln|x+4|]_1^b = \lim_{b\to\infty} \ln\frac{(b+2)^a}{b+4} - \ln\left(\frac{3^a}{5}\right);$
$\lim_{b\to\infty} \frac{(b+2)^a}{b+4} = a\lim_{b\to\infty} (b+2)^{a-1} = \begin{cases} \infty, & a > 1 \\ 1, & a = 1 \end{cases} \Rightarrow$ the series converges to $\ln\left(\frac{5}{3}\right)$ if $a = 1$ and diverges to ∞ if $a > 1$. If $a < 1$, the terms of the series eventually become negative and the Integral Test does not apply. From that point on, however, the series behaves like a negative multiple of the harmonic series, and so it diverges.

42. $\int_3^\infty \left(\frac{1}{x-1} - \frac{2a}{x+1}\right) dx = \lim_{b\to\infty} \left[\ln\left|\frac{x-1}{(x+1)^{2a}}\right|\right]_3^b = \lim_{b\to\infty} \ln\frac{b-1}{(b+1)^{2a}} - \ln\left(\frac{2}{4^{2a}}\right);\ \lim_{b\to\infty} \frac{b-1}{(b+1)^{2a}}$
$= \lim_{b\to\infty} \frac{1}{2a(b+1)^{2a-1}} = \begin{cases} 1, & a = \frac{1}{2} \\ \infty, & a < \frac{1}{2} \end{cases} \Rightarrow$ the series converges to $\ln\left(\frac{4}{2}\right) = \ln 2$ if $a = \frac{1}{2}$ and diverges to ∞ if

if $a < \frac{1}{2}$. If $a > \frac{1}{2}$, the terms of the series eventually become negative and the Integral Test does not apply. From that point on, however, the series behaves like a negative multiple of the harmonic series, and so it diverges.

43. (a)





(b) There are $(13)(365)(24)(60)(60)\left(10^9\right)$ seconds in 13 billion years; by part (a) $s_n \le 1 + \ln n$ where $n = (13)(365)(24)(60)(60)\left(10^9\right) \Rightarrow s_n \le 1 + \ln\left((13)(365)(24)(60)(60)\left(10^9\right)\right)$
$= 1 + \ln(13) + \ln(365) + \ln(24) + 2\ln(60) + 9\ln(10) \approx 41.55$

44. No, because $\sum_{n=1}^{\infty} \frac{1}{nx} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

45. Yes. If $\sum_{n=1}^{\infty} a_n$ is a divergent series of positive numbers, then $\left(\frac{1}{2}\right)\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{a_n}{2}\right)$ also diverges and $\frac{a_n}{2} < a_n$.

There is no "smallest" divergent series of positive numbers: for any divergent series $\sum_{n=1}^{\infty} a_n$ of positive numbers $\sum_{n=1}^{\infty} \left(\frac{a_n}{2}\right)$ has smaller terms and still diverges.

46. No, if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive numbers, then $2\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 2a_n$ also converges, and $2a_n \ge a_n$.
There is no "largest" convergent series of positive numbers.

47. (a) Both integrals can represent the area under the curve $f(x) = \frac{1}{\sqrt{x+1}}$, and the sum s_{50} can be considered an approximation of either integral using rectangles with $\Delta x = 1$. The sum $s_{50} = \sum_{n=1}^{50} \frac{1}{\sqrt{n+1}}$ is an overestimate of the integral $\int_1^{51} \frac{1}{\sqrt{x+1}}dx$. The sum s_{50} represents a left-hand sum (that is, the we are choosing the left-hand endpoint of each subinterval for c_i) and because f is a decreasing function, the value of f is a maximum at the left-hand endpoint of each sub interval. The area of each rectangle overestimates the true area, thus $\int_1^{51} \frac{1}{\sqrt{x+1}}dx < \sum_{n=1}^{50} \frac{1}{\sqrt{n+1}}$. In a similar manner, s_{50} underestimates the integral $\int_0^{50} \frac{1}{\sqrt{x+1}}dx$. In this case, the sum s_{50} represents a right-hand sum and because f is a decreasing function, the value of f is aminimum at the right-hand endpoint of each subinterval. The area of each rectangle underestimates the true area, thus $\sum_{n=1}^{50} \frac{1}{\sqrt{n+1}} < \int_0^{50} \frac{1}{\sqrt{x+1}}dx$. Evaluating the integrals we find $\int_1^{51} \frac{1}{\sqrt{x+1}}dx$
$= \left[2\sqrt{x+1}\right]_1^{51} = 2\sqrt{52} - 2\sqrt{2} \approx 11.6$ and $\int_0^{50} \frac{1}{\sqrt{x+1}}dx = \left[2\sqrt{x+1}\right]_0^{50} = 2\sqrt{51} - 2\sqrt{1} \approx 12.3$. Thus,
$11.6 < \sum_{n=1}^{50} \frac{1}{\sqrt{n+1}} < 12.3$.

(b) $s_n > 1000 \Rightarrow \int_1^{n+1} \frac{1}{\sqrt{x+1}}dx = \left[2\sqrt{x+1}\right]_1^{n+1} = 2\sqrt{n+1} - 2\sqrt{2} > 1000 \Rightarrow n > \left(500 + 2\sqrt{2}\right)^2 - \approx 251414.2$
$\Rightarrow n \ge 251415$.

48. (a) Since we are using $s_{30} = \sum_{n=1}^{30} \frac{1}{n^4}$ to estimate $\sum_{n=1}^{\infty} \frac{1}{n^4}$, the error is given by $\sum_{n=31}^{\infty} \frac{1}{n^4}$. We can consider this sum as an estimate of the area under the curve $f(x) = \frac{1}{x^4}$ when $x \ge 30$ using rectangles with $\Delta x = 1$ and c_i is the right-hand endpoint of each subinterval. Since f is a decreasing function, the value of f is a minimum at the right-hand endpoint of each subinterval, thus $\sum_{n=31}^{\infty} \frac{1}{n^4} < \int_{30}^{\infty} \frac{1}{x^4}dx = \lim_{b\to\infty} \int_{30}^{b} \frac{1}{x^4}dx = \lim_{b\to\infty} \left[-\frac{1}{3x^3}\right]_{30}^{b} = \lim_{b\to\infty} \left(-\frac{1}{3b^3} + \frac{1}{3(30)^3}\right) \approx 1.23 \times 10^{-5}$. Thus the error $< 1.23 \times 10^{-5}$.

(b) We want $S - s_n < 0.000001 \Rightarrow \int_{n}^{\infty} \frac{1}{x^4}dx < 0.000001 \Rightarrow \int_{n}^{\infty} \frac{1}{x^4}dx = \lim_{b\to\infty} \int_{n}^{b} \frac{1}{x^4}dx = \lim_{b\to\infty} \left[-\frac{1}{3x^3}\right]_{n}^{b}$ $= \lim_{b\to\infty} \left(-\frac{1}{3b^3} + \frac{1}{3n^3}\right) = \frac{1}{3n^3} < 0.000001 \Rightarrow n > \sqrt[3]{\frac{1000000}{3}} \approx 69.336 \Rightarrow n \ge 70.$

49. We want $S - s_n < 0.01 \Rightarrow \int_{n}^{\infty} \frac{1}{x^3}dx < 0.01 \Rightarrow \int_{n}^{\infty} \frac{1}{x^3}dx = \lim_{b\to\infty} \int_{n}^{b} \frac{1}{x^3}dx = \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_{n}^{b} = \lim_{b\to\infty} \left(-\frac{1}{2b^2} + \frac{1}{2n^2}\right)$ $= \frac{1}{2n^2} < 0.01 \Rightarrow n > \sqrt{50} \approx 7.071 \Rightarrow n \ge 8 \Rightarrow S \approx s_8 = \sum_{n=1}^{8} \frac{1}{n^3} \approx 1.195$

50. We want $S - s_n < 0.1 \Rightarrow \int_{n}^{\infty} \frac{1}{x^2+4}dx < 0.1 \Rightarrow \lim_{b\to\infty} \int_{n}^{b} \frac{1}{x^2+4}dx = \lim_{b\to\infty} \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_{n}^{b}$ $= \lim_{b\to\infty} \left(\frac{1}{2}\tan^{-1}\left(\frac{b}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{n}{2}\right)\right) = \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{n}{2}\right) < 0.1 \Rightarrow n > 2\tan\left(\frac{\pi}{2} - 0.2\right) \approx 9.867 \Rightarrow n \ge 10 \Rightarrow S \approx s_{10}$ $= \sum_{n=1}^{10} \frac{1}{n^2+4} \approx 0.57$

51. $S - s_n < 0.00001 \Rightarrow \int_{n}^{\infty} \frac{1}{x^{1.1}}dx < 0.00001 \Rightarrow \int_{n}^{\infty} \frac{1}{x^{1.1}}dx = \lim_{b\to\infty} \int_{n}^{b} \frac{1}{x^{1.1}}dx = \lim_{b\to\infty} \left[-\frac{10}{x^{0.1}}\right]_{n}^{b} = \lim_{b\to\infty} \left(-\frac{10}{b^{0.1}} + \frac{10}{n^{0.1}}\right)$ $= \frac{10}{n^{0.1}} < 0.00001 \Rightarrow n > 1000000^{10} \Rightarrow n > 10^{60}$

52. $S - s_n < 0.01 \Rightarrow \int_{n}^{\infty} \frac{1}{x(\ln x)^3}dx < 0.01 \Rightarrow \int_{n}^{\infty} \frac{1}{x(\ln x)^3}dx = \lim_{b\to\infty} \int_{n}^{b} \frac{1}{x(\ln x)^3}dx = \lim_{b\to\infty} \left[-\frac{1}{2(\ln x)^2}\right]_{n}^{b}$ $= \lim_{b\to\infty} \left(-\frac{1}{2(\ln b)^2} + \frac{1}{2(\ln n)^2}\right) = \frac{1}{2(\ln n)^2} < 0.01 \Rightarrow n > e^{\sqrt{50}} \approx 1177.405 \Rightarrow n \ge 1178$

53. Let $A_n = \sum_{k=1}^{n} a_k$ and $B_n = \sum_{k=1}^{n} 2^k a_{(2^k)}$, where $\{a_k\}$ is a nonincreasing sequence of positive terms converging to 0. Note that $\{A_n\}$ and $\{B_n\}$ are nondecreasing sequences of positive terms. Now,

$$B_n = 2a_2 + 4a_4 + 8a_8 + \ldots + 2^n a_{(2^n)} = 2a_2 + (2a_4 + 2a_4) + (2a_8 + 2a_8 + 2a_8 + 2a_8) + \ldots$$
$$+ \underbrace{\left(2a_{(2^n)} + 2a_{(2^n)} + \ldots + 2a_{(2^n)}\right)}_{2^{n-1}\text{ terms}} \le 2a_1 + 2a_2 + (2a_3 + 2a_4) + (2a_5 + 2a_6 + 2a_7 + 2a_8) + \ldots$$
$$+ \left(2a_{(2^{n-1})} + 2a_{(2^{n-1}+1)} + \ldots + 2a_{(2^n)}\right) = 2A_{(2^n)} \le 2\sum_{k=1}^{\infty} a_k.$$ Therefore if $\sum a_k$ converges, then $\{B_n\}$ is bounded above $\Rightarrow \sum 2^k a_{(2^k)}$ converges. Conversely,

$$A_n = a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \ldots + a_n < a_1 + 2a_2 + 4a_4 + \ldots + 2^n a_{(2^n)} = a_1 + B_n < a_1 + \sum_{k=1}^{\infty} 2^k a_{(2^k)}.$$

Therefore, if $\sum_{k=1}^{\infty} 2^k a_{(2^k)}$ converges, then $\{A_n\}$ is bounded above and hence converges.

54. (a) $a_{(2^n)} = \frac{1}{2^n \ln(2^n)} = \frac{1}{2^n \cdot n(\ln 2)} \Rightarrow \sum_{n=2}^{\infty} 2^n a_{(2^n)} = \sum_{n=2}^{\infty} 2^n \frac{1}{2^n \cdot n(\ln 2)} = \frac{1}{\ln 2} \sum_{n=2}^{\infty} \frac{1}{n}$, which diverges $\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

(b) $a_{(2^n)} = \frac{1}{2^{np}} \Rightarrow \sum_{n=1}^{\infty} 2^n a_{(2^n)} = \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^{np}} = \sum_{n=1}^{\infty} \frac{1}{(2^n)^{p-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n$, a geometric series that converges if $\frac{1}{2^{p-1}} < 1$ or $p > 1$, but diverges if $p \le 1$.

55. (a) $\int_2^{\infty} \frac{dx}{x(\ln x)^p}$; $\begin{bmatrix} u = \ln x \\ du = \frac{dx}{x} \end{bmatrix} \rightarrow \int_{\ln 2}^{\infty} u^{-p}\, du = \lim_{b \to \infty} \left[\frac{u^{-p+1}}{-p+1}\right]_{\ln 2}^{b} = \lim_{b \to \infty} \left(\frac{1}{1-p}\right)\left[b^{-p+1} - (\ln 2)^{-p+1}\right]$

$= \begin{cases} \frac{1}{p-1}(\ln 2)^{-p+1}, & p > 1 \\ \infty, & p < 1 \end{cases} \Rightarrow$ the improper integral converges if $p > 1$ and diverges if $p < 1$.

For $p = 1$: $\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{b \to \infty} \left[\ln(\ln x)\right]_2^b = \lim_{b \to \infty} \left[\ln(\ln b) - \ln(\ln 2)\right] = \infty$, so the improper integral diverges if $p = 1$.

(b) Since the series and the integral converge or diverge together, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges if and only if $p > 1$.

56. (a) $p = 1 \Rightarrow$ the series diverges

(b) $p = 1.01 \Rightarrow$ the series converges

(c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n^3)} = \frac{1}{3} \sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$; $p = 1 \Rightarrow$ the series diverges

(d) $p = 3 \Rightarrow$ the series converges

57. (a) From Fig. 10.11(a) in the text with $f(x) = \frac{1}{x}$ and $a_k = \frac{1}{k}$, we have $\int_1^{n+1} \frac{1}{x}\, dx \le 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$

$\le 1 + \int_1^n f(x)\, dx \Rightarrow \ln(n+1) \le 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \le 1 + \ln n \Rightarrow 0 \le \ln(n+1) - \ln n$

$\le \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) - \ln n \le 1$. Therefore the sequence $\left\{\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) - \ln n\right\}$ is bounded above by 1 and below by 0.

(b) From the graph in Fig. 10.11(b) with $f(x) = \frac{1}{x}$, $\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x}\, dx = \ln(n+1) - \ln n$

$\Rightarrow 0 > \frac{1}{n+1} - [\ln(n+1) - \ln n] = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n+1} - \ln(n+1)\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n\right)$.

If we define $a_n = 1 + \frac{1}{2} = \frac{1}{3} + \frac{1}{n} - \ln n$, then $0 > a_{n+1} - a_n \Rightarrow a_{n+1} < a_n \Rightarrow \{a_n\}$ is a decreasing sequence of nonnegative terms.

58. $e^{-x^2} \le e^{-x}$ for $x \ge 1$, and $\int_1^{\infty} e^{-x}\, dx = \lim_{b \to \infty} \left[-e^{-x}\right]_1^b = \lim_{b \to \infty} \left(-e^{-b} + e^{-1}\right) = e^{-1} \Rightarrow \int_1^{\infty} e^{-x^2}\, dx$ converges by the Comparison Test for improper integrals $\Rightarrow \sum_{n=0}^{\infty} e^{-n^2} = 1 + \sum_{n=1}^{\infty} e^{-n^2}$ converges by the Integral Test.

59. (a) $s_{10} = \sum_{n=1}^{10} \frac{1}{n^3} = 1.97531986$; $\int_{11}^{\infty} \frac{1}{x^3}\, dx = \lim_{b \to \infty} \int_{11}^{b} x^{-3}\, dx = \lim_{b \to \infty} \left[-\frac{x^{-2}}{2}\right]_{11}^{b} = \lim_{b \to \infty} \left(-\frac{1}{2b^2} + \frac{1}{242}\right) = \frac{1}{242}$ and

$\int_{10}^{\infty} \frac{1}{x^3}\, dx = \lim_{b \to \infty} \int_{10}^{b} x^{-3}\, dx = \lim_{b \to \infty} \left[-\frac{x^{-2}}{2}\right]_{10}^{b} = \lim_{b \to \infty} \left(-\frac{1}{2b^2} + \frac{1}{200}\right) = \frac{1}{200}$

$\Rightarrow 1.97531986 + \frac{1}{242} < s < 1.97531986 + \frac{1}{200} \Rightarrow 1.20166 < s < 1.20253$

(b) $s = \sum_{n=1}^{\infty} \frac{1}{n^3} \approx \frac{1.20166 + 1.20253}{2} = 1.202095$; error $\le \frac{1.20253 - 1.20166}{2} = 0.000435$

60. (a) $s_{10} = \sum_{n=1}^{10} \frac{1}{n^4} = 1.082036583$; $\int_{11}^{\infty} \frac{1}{x^4}\, dx = \lim_{b \to \infty} \int_{11}^{b} x^{-4}\, dx = \lim_{b \to \infty} \left[-\frac{x^{-3}}{3}\right]_{11}^{b} = \lim_{b \to \infty} \left(-\frac{1}{3b^3} + \frac{1}{3993}\right) = \frac{1}{3993}$ and

$\int_{10}^{\infty} \frac{1}{x^4}\, dx = \lim_{b \to \infty} \int_{10}^{b} x^{-4}\, dx = \lim_{b \to \infty} \left[-\frac{x^{-3}}{3}\right]_{10}^{b} = \lim_{b \to \infty} \left(-\frac{1}{3b^3} + \frac{1}{3000}\right) = \frac{1}{3000}$

$\Rightarrow 1.082036583 + \frac{1}{3993} < s < 1.082036583 + \frac{1}{3000} \Rightarrow 1.08229 < s < 1.08237$

(b) $s = \sum_{n=1}^{\infty} \frac{1}{n^4} \approx \frac{1.08229 + 1.08237}{2} = 1.08233$; error $\le \frac{1.08237 - 1.08229}{2} = 0.00004$

10.4 COMPARISON TESTS

1. Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series, since $p = 2 > 1$. Both series have nonnegative terms for $n \geq 1$. For $n \geq 1$, we have $n^2 \leq n^2 + 30 \Rightarrow \frac{1}{n^2} \geq \frac{1}{n^2+30}$. Then by Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n^2+30}$ converges.

2. Compare with $\sum_{n=1}^{\infty} \frac{1}{n^3}$, which is a convergent p-series, since $p = 3 > 1$. Both series have nonnegative terms for $n \geq 1$. For $n \geq 1$, we have $n^4 \leq n^4 + 2 \Rightarrow \frac{1}{n^4} \geq \frac{1}{n^4+2} \Rightarrow \frac{n}{n^4} \geq \frac{n}{n^4+2} \Rightarrow \frac{1}{n^3} \geq \frac{n}{n^4+2} \geq \frac{n-1}{n^4+2}$. Then by Comparison Test, $\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$ converges.

3. Compare with $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series, since $p = \frac{1}{2} \leq 1$. Both series have nonnegative terms for $n \geq 2$. For $n \geq 2$, we have $\sqrt{n} - 1 \leq \sqrt{n} \Rightarrow \frac{1}{\sqrt{n}-1} \geq \frac{1}{\sqrt{n}}$. Then by Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges.

4. Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is a divergent p-series, since $p = 1 \leq 1$. Both series have nonnegative terms for $n \geq 2$. For $n \geq 2$, we have $n^2 - n \leq n^2 \Rightarrow \frac{1}{n^2-n} \geq \frac{1}{n^2} \Rightarrow \frac{n}{n^2-n} \geq \frac{n}{n^2} = \frac{1}{n} \Rightarrow \frac{n+2}{n^2-n} \geq \frac{n}{n^2-n} \geq \frac{1}{n}$. Thus $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ diverges.

5. Compare with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which is a convergent p-series, since $p = \frac{3}{2} > 1$. Both series have nonnegative terms for $n \geq 1$. For $n \geq 1$, we have $0 \leq \cos^2 n \leq 1 \Rightarrow \frac{\cos^2 n}{n^{3/2}} \leq \frac{1}{n^{3/2}}$. Then by Comparison Test, $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$ converges.

6. Compare with $\sum_{n=1}^{\infty} \frac{1}{3^n}$, which is a convergent geometric series, since $|r| = \left|\frac{1}{3}\right| < 1$. Both series have nonnegative terms for $n \geq 1$. For $n \geq 1$, we have $n \cdot 3^n \geq 3^n \Rightarrow \frac{1}{n \cdot 3^n} \leq \frac{1}{3^n}$. Then by Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n}$ converges.

7. Compare with $\sum_{n=1}^{\infty} \frac{\sqrt{5}}{n^{3/2}}$. The series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series, since $p = \frac{3}{2} > 1$, and the series $\sum_{n=1}^{\infty} \frac{\sqrt{5}}{n^{3/2}}$
$= \sqrt{5}\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by Theorem 8 part 3. Both series have nonnegative terms for $n \geq 1$. For $n \geq 1$, we have
$n^3 \leq n^4 \Rightarrow 4n^3 \leq 4n^4 \Rightarrow n^4 + 4n^3 \leq n^4 + 4n^4 = 5n^4 \Rightarrow n^4 + 4n^3 \leq 5n^4 + 20 = 5(n^4 + 4) \Rightarrow \frac{n^4+4n^3}{n^4+4} \leq 5.$
$\Rightarrow \frac{n^3(n+4)}{n^4+4} \leq 5 \Rightarrow \frac{n+4}{n^4+4} \leq \frac{5}{n^3} \Rightarrow \sqrt{\frac{n+4}{n^4+4}} \leq \sqrt{\frac{5}{n^3}} = \frac{\sqrt{5}}{n^{3/2}}$ Then by Comparison Test, $\sum_{n=1}^{\infty} \sqrt{\frac{n+4}{n^4+4}}$ converges.

8. Compare with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series, since $p = \frac{1}{2} \leq 1$. Both series have nonnegative terms for $n \geq 1$. For $n \geq 1$, we have $\sqrt{n} \geq 1 \Rightarrow 2\sqrt{n} \geq 2 \Rightarrow 2\sqrt{n} + 1 \geq 3 \Rightarrow n(2\sqrt{n} + 1) \geq 3n \geq 3 \Rightarrow 2n\sqrt{n} + n \geq 3$
$\Rightarrow n^2 + 2n\sqrt{n} + n \geq n^2 + 3 \Rightarrow \frac{n(n+2\sqrt{n}+1)}{n^2+3} \geq 1 \Rightarrow \frac{n+2\sqrt{n}+1}{n^2+3} \geq \frac{1}{n} \Rightarrow \frac{(\sqrt{n}+1)^2}{n^2+3} \geq \frac{1}{n} \Rightarrow \sqrt{\frac{(\sqrt{n}+1)^2}{n^2+3}} \geq \sqrt{\frac{1}{n}}$
$\Rightarrow \frac{\sqrt{n}+1}{\sqrt{n^2+3}} \geq \frac{1}{\sqrt{n}}$. Then by Comparison Test, $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$ diverges.

9. Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series, since $p = 2 > 1$. Both series have positive terms for $n \geq 1$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n-2}{n^3-n^2+3}}{1/n^2} = \lim_{n\to\infty} \frac{n^3-2n^2}{n^3-n^2+3} = \lim_{n\to\infty} \frac{3n^2-4n}{3n^2-2n} = \lim_{n\to\infty} \frac{6n-4}{6n-2} = \lim_{n\to\infty} \frac{6}{6} = 1 > 0$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{n-2}{n^3-n^2+3}$ converges.

10. Compare with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series, since $p = \frac{1}{2} \leq 1$. Both series have positive terms for $n \geq 1$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{\frac{n+1}{n^2+2}}}{1/\sqrt{n}} = \lim_{n\to\infty} \sqrt{\frac{n^2+n}{n^2+2}} = \sqrt{\lim_{n\to\infty} \frac{n^2+n}{n^2+2}} = \sqrt{\lim_{n\to\infty} \frac{2n+1}{2n}} = \sqrt{\lim_{n\to\infty} \frac{2}{2}} = \sqrt{1} = 1 > 0$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+2}}$ diverges.

11. Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is a divergent p-series, since $p = 1 \leq 1$. Both series have positive terms for $n \geq 2$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n(n+1)}{(n^2+1)(n-1)}}{1/n} = \lim_{n\to\infty} \frac{n^3+n^2}{n^3-n^2+n-1} = \lim_{n\to\infty} \frac{3n^2+2n}{3n^2-2n+1} = \lim_{n\to\infty} \frac{6n+2}{6n-2} = \lim_{n\to\infty} \frac{6}{6} = 1 > 0$. Then by Limit Comparison Test, $\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)}$ diverges.

12. Compare with $\sum_{n=1}^{\infty} \frac{1}{2^n}$, which is a convergent geometric series, since $|r| = \left|\frac{1}{2}\right| < 1$. Both series have positive terms for $n \geq 1$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2^n}{3+4^n}}{1/2^n} = \lim_{n\to\infty} \frac{4^n}{3+4^n} = \lim_{n\to\infty} \frac{4^n \ln 4}{4^n \ln 4} = 1 > 0$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$ converges.

13. Compare with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series, since $p = \frac{1}{2} \leq 1$. Both series have positive terms for $n \geq 1$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{5^n}{\sqrt{n}\cdot 4^n}}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{5^n}{4^n} = \lim_{n\to\infty} \left(\frac{5}{4}\right)^n = \infty$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\cdot 4^n}$ diverges.

14. Compare with $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$, which is a convergent geometric series, since $|r| = \left|\frac{2}{5}\right| < 1$. Both series have positive terms for $n \geq 1$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{2n+3}{5n+4}\right)^n}{(2/5)^n} = \lim_{n\to\infty} \left(\frac{10n+15}{10n+8}\right)^n = \exp \lim_{n\to\infty} \ln\left(\frac{10n+15}{10n+8}\right)^n = \exp \lim_{n\to\infty} n \ln\left(\frac{10n+15}{10n+8}\right)$
$= \exp \lim_{n\to\infty} \frac{\ln\left(\frac{10n+15}{10n+8}\right)}{1/n} = \exp \lim_{n\to\infty} \frac{\frac{10}{10n+15} - \frac{10}{10n+8}}{-1/n^2} = \exp \lim_{n\to\infty} \frac{70n^2}{(10n+15)(10n+8)} = \exp \lim_{n\to\infty} \frac{70n^2}{100n^2+230n+120}$
$= \exp \lim_{n\to\infty} \frac{140n}{200n+230} = \exp \lim_{n\to\infty} \frac{140}{200} = e^{7/10} > 0$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$ converges.

15. Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is a divergent p-series, since $p = 1 \leq 1$. Both series have positive terms for $n \geq 2$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\ln n}}{1/n} = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{n\to\infty} \frac{1}{1/n} = \lim_{n\to\infty} n = \infty$. Then by Limit Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges.

16. Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series, since $p = 2 > 1$. Both series have positive terms for $n \geq 1$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{1/n^2} = \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n^2}}\left(-\frac{2}{n^3}\right)}{\left(-\frac{2}{n^3}\right)} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1 > 0$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right)$ converges.

17. diverges by the Limit Comparison Test (part 1) when compared with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, a divergent p-series:

$$\lim_{n\to\infty} \frac{\left(\frac{1}{2\sqrt{n}+\sqrt[3]{n}}\right)}{\left(\frac{1}{\sqrt{n}}\right)} = \lim_{n\to\infty} \frac{\sqrt{n}}{2\sqrt{n}+\sqrt[3]{n}} = \lim_{n\to\infty} \left(\frac{1}{2+n^{-1/6}}\right) = \frac{1}{2}$$

18. diverges by the Direct Comparison Test since $n+n+n > n+\sqrt{n}+0 \Rightarrow \frac{3}{n+\sqrt{n}} > \frac{1}{n}$, which is the nth term of the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$ or use Limit Comparison Test with $b_n = \frac{1}{n}$

19. converges by the Direct Comparison Test; $\frac{\sin^2 n}{2^n} \le \frac{1}{2^n}$, which is the nth term of a convergent geometric series

20. converges by the Direct Comparison Test; $\frac{1+\cos n}{n^2} \le \frac{2}{n^2}$ and the p-series $\sum \frac{1}{n^2}$ converges

21. diverges since $\lim_{n\to\infty} \frac{2n}{3n-1} = \frac{2}{3} \ne 0$

22. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^{3/2}}$, the nth term of a convergent p-series:

$$\lim_{n\to\infty} \frac{\left(\frac{n+1}{n^2\sqrt{n}}\right)}{\left(\frac{1}{n^{3/2}}\right)} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right) = 1$$

23. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$, the nth term of a convergent p-series:

$$\lim_{n\to\infty} \frac{\left(\frac{10n+1}{n(n+1)(n+2)}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{10n^2+n}{n^2+3n+2} = \lim_{n\to\infty} \frac{20n+1}{2n+3} = \lim_{n\to\infty} \frac{20}{2} = 10$$

24. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$, the nth term of a convergent p-series:

$$\lim_{n\to\infty} \frac{\left(\frac{5n^3-3n}{n^2(n-2)\left(n^2+5\right)}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{5n^3-3n}{n^3-2n^2+5n-10} = \lim_{n\to\infty} \frac{15n^2-3}{3n^2-4n+5} = \lim_{n\to\infty} \frac{30n}{6n-4} = 5$$

25. converges by the Direct Comparison Test; $\left(\frac{n}{3n+1}\right)^n < \left(\frac{n}{3n}\right)^n = \left(\frac{1}{3}\right)^n$, the nth term of a convergent geometric series

26. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^{3/2}}$, the nth term of a convergent p-series:

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n^{3/2}}\right)}{\left(\frac{1}{\sqrt{n^3+2}}\right)} = \lim_{n\to\infty} \sqrt{\frac{n^3+2}{n^3}} = \lim_{n\to\infty} \sqrt{1+\frac{2}{n^3}} = 1$$

27. diverges by the Direct Comparison Test; $n > \ln n \Rightarrow \ln n > \ln \ln n \Rightarrow \frac{1}{n} < \frac{1}{\ln n} < \frac{1}{\ln(\ln n)}$ and $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges

28. converges by the Limit Comparison Test (part 2) when compared with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a convergent p-series:

$$\lim_{n\to\infty} \frac{\left[\frac{(\ln n)^2}{n^3}\right]}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{(\ln n)^2}{n} = \lim_{n\to\infty} \frac{2(\ln n)\left(\frac{1}{n}\right)}{1} = 2 \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

29. diverges by the Limit Comparison Test (part 3) with $\frac{1}{n}$, the nth term of the divergent harmonic series:

$$\lim_{n\to\infty} \frac{\left[\frac{1}{\sqrt{n}\ln n}\right]}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{\sqrt{n}}{\ln n} = \lim_{n\to\infty} \frac{\left(\frac{1}{2\sqrt{n}}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} = \infty$$

30. converges by the Limit Comparison Test (part 2) with $\frac{1}{n^{5/4}}$, the nth term of a convergent p-series:
$$\lim_{n \to \infty} \frac{\left[\frac{(\ln n)^2}{n^{3/2}}\right]}{\left(\frac{1}{n^{5/4}}\right)} = \lim_{n \to \infty} \frac{(\ln n)^2}{n^{1/4}} = \lim_{n \to \infty} \frac{\left(\frac{2 \ln n}{n}\right)}{\left(\frac{1}{4n^{3/4}}\right)} = 8 \lim_{n \to \infty} \frac{\ln n}{n^{1/4}} = 8 \lim_{n \to \infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{4n^{3/4}}\right)} = 32 \lim_{n \to \infty} \frac{1}{n^{1/4}} = 32 \cdot 0 = 0$$

31. diverges by the Limit Comparison Test (part 3) with $\frac{1}{n}$, the nth term of the divergent harmonic series:
$$\lim_{n \to \infty} \frac{\left(\frac{1}{1+\ln n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \to \infty} \frac{n}{1+\ln n} = \lim_{n \to \infty} \frac{1}{\left(\frac{1}{n}\right)} = \lim_{n \to \infty} n = \infty$$

32. diverges by the Integral Test: $\int_2^\infty \frac{\ln(x+1)}{x+1}\,dx = \int_{\ln 3}^\infty u\,du = \lim_{b \to \infty} \left[\frac{1}{2}u^2\right]_{\ln 3}^{b} = \lim_{b \to \infty} \frac{1}{2}\left(b^2 - \ln^2 3\right) = \infty$

33. converges by the Direct Comparison Test with $\frac{1}{n^{3/2}}$, the nth term of a convergent p-series: $n^2 - 1 > n$ for $n \ge 2 \Rightarrow n^2\left(n^2 - 1\right) > n^3 \Rightarrow n\sqrt{n^2 - 1} > n^{3/2} \Rightarrow \frac{1}{n^{3/2}} > \frac{1}{n\sqrt{n^2-1}}$ or use Limit Comparison Test with $\frac{1}{n^2}$.

34. converges by the Direct Comparison Test with $\frac{1}{n^{3/2}}$, the nth term of a convergent p-series: $n^2 + 1 > n^2$ $\Rightarrow n^2 + 1 > \sqrt{n}n^{3/2} \Rightarrow \frac{n^2+1}{\sqrt{n}} > n^{3/2} \Rightarrow \frac{\sqrt{n}}{n^2+1} < \frac{1}{n^{3/2}}$ or use Limit Comparison Test with $\frac{1}{n^{3/2}}$.

35. converges because $\sum_{n=1}^{\infty} \frac{1-n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n2^n} + \sum_{n=1}^{\infty} \frac{-1}{2^n}$ which is the sum of two convergent series: $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converges by the Direct Comparison Test since $\frac{1}{n2^n} < \frac{1}{2^n}$, and $\sum_{n=1}^{\infty} \frac{-1}{2^n}$ is a convergent geometric series

36. converges by the Direct Comparison Test: $\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{n2^n} + \frac{1}{n^2}\right)$ and $\frac{1}{n2^n} + \frac{1}{n^2} \le \frac{1}{2^n} + \frac{1}{n^2}$, the sum of the nth terms of a convergent geometric series and a convergent p-series

37. converges by the Direct Comparison Test: $\frac{1}{3^{n-1}+1} < \frac{1}{3^{n-1}}$, which is the nth term of a convergent geometric series

38. diverges; $\lim_{n \to \infty} \left(\frac{3^{n-1}+1}{3^n}\right) = \lim_{n \to \infty} \left(\frac{1}{3} + \frac{1}{3^n}\right) = \frac{1}{3} \ne 0$

39. converges by Limit Comparison Test: compare with $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$, which is a convergent geometric series with $|r| = \frac{1}{5} < 1$,
$$\lim_{n \to \infty} \frac{\left(\frac{n+1}{n^2+3n} \cdot \frac{1}{5^n}\right)}{(1/5)^n} = \lim_{n \to \infty} \frac{n+1}{n^2+3n} = \lim_{n \to \infty} \frac{1}{2n+3} = 0.$$

40. converges by Limit Comparison Test: compare with $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$, which is a convergent geometric series with $|r| = \frac{1}{5} < 1$,
$$\lim_{n \to \infty} \frac{\left(\frac{2^n+3^n}{3^n+4^n}\right)}{(3/4)^n} = \lim_{n \to \infty} \frac{8^n+12^n}{9^n+12^n} = \lim_{n \to \infty} \frac{\left(\frac{8}{12}\right)^n+1}{\left(\frac{9}{12}\right)^n+1} = \frac{1}{1} = 1 > 0.$$

41. diverges by Limit Comparison Test: compare with $\sum_{n=1}^{\infty} \frac{1}{n}$, which is a divergent p-series, $\lim_{n \to \infty} \frac{\left(\frac{2^n-n}{n \cdot 2^n}\right)}{1/n} = \lim_{n \to \infty} \frac{2^n-n}{2^n}$ $= \lim_{n \to \infty} \frac{2^n \ln 2 - 1}{2^n \ln 2} = \lim_{n \to \infty} \frac{2^n (\ln 2)^2}{2^n (\ln 2)^2} = 1 > 0.$

42. diverges by the definition of an infinite series: $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} \left[\ln n - \ln(n+1)\right]$, $s_k = (\ln 1 - \ln 2) + (\ln 2 - \ln 3)$ $+ \ldots + (\ln(k-1) - \ln k) + (\ln k - \ln(k+1)) = -\ln(k+1) \Rightarrow \lim_{k \to \infty} s_k = -\infty$

43. converges by Comparison Test with $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ which converges since $\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty} \left[\frac{1}{n-1} - \frac{1}{n}\right]$, and $s_k = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{k-2} - \frac{1}{k-1}\right) + \left(\frac{1}{k-1} - \frac{1}{k}\right) = 1 - \frac{1}{k} \Rightarrow \lim_{k \to \infty} s_k = 1$; for $n \ge 2$, $(n-2)! \ge 1$ $\Rightarrow n(n-1)(n-2)! \ge n(n-1) \Rightarrow n! \ge n(n-1) \Rightarrow \frac{1}{n!} \le \frac{1}{n(n-1)}$

44. converges by Limit Comparison Test: compare with $\sum_{n=1}^{\infty} \frac{1}{n^3}$, which is a convergent p-series, $\lim_{n \to \infty} \frac{\frac{(n-1)!}{(n+2)!}}{1/n^3}$ $= \lim_{n \to \infty} \frac{n^3(n-1)!}{(n+2)(n+1)n(n-1)!} = \lim_{n \to \infty} \frac{n^2}{n^2+3n+2} = \lim_{n \to \infty} \frac{2n}{2n+3} = \lim_{n \to \infty} \frac{2}{2} = 1 > 0$

45. diverges by the Limit Comparison Test (part 1) with $\frac{1}{n}$, the nth term of the divergent harmonic series: $\lim_{n \to \infty} \frac{\left(\sin \frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{x \to 0} \frac{\sin x}{x} = 1$

46. diverges by the Limit Comparison Test (part 1) with $\frac{1}{n}$, the nth term of the divergent harmonic series: $\lim_{n \to \infty} \frac{\left(\tan \frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \to \infty} \left(\frac{1}{\cos \frac{1}{n}}\right) \frac{\left(\sin \frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{x \to 0} \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{x}\right) = 1 \cdot 1 = 1$

47. converges by the Direct Comparison Test: $\frac{\tan^{-1} n}{n^{1.1}} < \frac{\frac{\pi}{2}}{n^{1.1}}$ and $\sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^{1.1}} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ is the product of a convergent p-series and a nonzero constant

48. converges by the Direct Comparison Test: $\sec^{-1} n < \frac{\pi}{2} \Rightarrow \frac{\sec^{-1} n}{n^{1.3}} < \frac{\left(\frac{\pi}{2}\right)}{n^{1.3}}$ and $\sum_{n=1}^{\infty} \frac{\left(\frac{\pi}{2}\right)}{n^{1.3}} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^{1.3}}$ is the product of a convergent p-series and a nonzero constant

49. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$: $\lim_{n \to \infty} \frac{\left(\frac{\coth n}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n \to \infty} \coth n = \lim_{n \to \infty} \frac{e^n + e^{-n}}{e^n - e^{-n}}$ $= \lim_{n \to \infty} \frac{1 + e^{-2n}}{1 - e^{-2n}} = 1$

50. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$: $\lim_{n \to \infty} \frac{\left(\frac{\tanh n}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n \to \infty} \tanh n = \lim_{n \to \infty} \frac{e^n - e^{-n}}{e^n + e^{-n}}$ $= \lim_{n \to \infty} \frac{1 - e^{-2n}}{1 + e^{-2n}} = 1$

51. diverges by the Limit Comparison Test (part 1) with $\frac{1}{n}$: $\lim_{n \to \infty} \frac{\left(\frac{1}{n\sqrt[n]{n}}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \to \infty} \frac{1}{\sqrt[n]{n}} = 1.$

52. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$: $\lim_{n \to \infty} \frac{\left(\frac{\sqrt[n]{n}}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n \to \infty} \sqrt[n]{n} = 1$

53. $\frac{1}{1+2+3+\ldots+n} = \frac{1}{\left(\frac{n(n+1)}{2}\right)} = \frac{2}{n(n+1)}$. The series converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$: $\lim_{n \to \infty} \frac{\left(\frac{2}{n(n+1)}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n \to \infty} \frac{2n^2}{n^2+n} = \lim_{n \to \infty} \frac{4n}{2n+1} = \lim_{n \to \infty} \frac{4}{2} = 2.$

54. $\frac{1}{1+2^2+3^2+\ldots+n^2} = \frac{1}{\frac{n(n+1)(2n+1)}{6}} = \frac{6}{n(n+1)(2n+1)} \le \frac{6}{n^3} \Rightarrow$ the series converges by the Direct Comparison Test

55. (a) If $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$, then there exists an integer N such that for all $n > N$, $\left|\frac{a_n}{b_n} - 0\right| < 1 \Rightarrow -1 < \frac{a_n}{b_n} < 1$ $\Rightarrow a_n < b_n$. Thus, if $\sum b_n$ converges, then $\sum a_n$ converges by the Direct Comparison Test.

(b) If $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$, then there exists an integer N such that for all $n > N$, $\frac{a_n}{b_n} > 1 \Rightarrow a_n > b_n$. Thus, if $\sum b_n$ diverges, then $\sum a_n$ diverges by the Direct Comparison Test.

56. Yes, $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges by the Direct Comparison Test because $\frac{a_n}{n} < a_n$

57. $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty \Rightarrow$ there exists an integer N such that for all $n > N$, $\frac{a_n}{b_n} > 1 \Rightarrow a_n > b_n$. If $\sum a_n$ converges, then $\sum b_n$ converges by the Direct Comparison Test

58. $\sum a_n$ converges $\Rightarrow \lim_{n \to \infty} a_n = 0 \Rightarrow$ there exists an integer N such that for all $n > N$, $0 \le a_n < 1 \Rightarrow a_n^2 < a_n$ $\Rightarrow \sum a_n^2$ converges by the Direct Comparison Test

59. Since $a_n > 0$ and $\lim_{n \to \infty} a_n = \infty \ne 0$, by n^{th} term test for divergence, $\sum a_n$ diverges.

60. Since $a_n > 0$ and $\lim_{n \to \infty} (n^2 \cdot a_n) = 0$, compare $\sum a_n$ with $\sum \frac{1}{n^2}$, which is a convergent p-series; $\lim_{n \to \infty} \frac{a_n}{1/n^2}$ $= \lim_{n \to \infty} (n^2 \cdot a_n) = 0 \Rightarrow \sum a_n$ converges by Limit Comparison Test

61. Let $-\infty < q < \infty$ and $p > 1$. If $q = 0$, then $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p} = \sum_{n=2}^{\infty} \frac{1}{n^p}$, which is a convergent p-series. If $q \ne 0$, compare with $\sum_{n=2}^{\infty} \frac{1}{n^r}$ where $1 < r < p$, then $\lim_{n \to \infty} \frac{\frac{(\ln n)^q}{n^p}}{1/n^r} = \lim_{n \to \infty} \frac{(\ln n)^q}{n^{p-r}}$, and $p - r > 0$. If $q < 0 \Rightarrow -q > 0$ and $\lim_{n \to \infty} \frac{(\ln n)^q}{n^{p-r}}$ $= \lim_{n \to \infty} \frac{1}{(\ln n)^{-q} n^{p-r}} = 0$. If $q > 0$, $\lim_{n \to \infty} \frac{(\ln n)^q}{n^{p-r}} = \lim_{n \to \infty} \frac{q(\ln n)^{q-1}\left(\frac{1}{n}\right)}{(p-r)n^{p-r-1}} = \lim_{n \to \infty} \frac{q(\ln n)^{q-1}}{(p-r)n^{p-r}}$. If $q - 1 \le 0 \Rightarrow 1 - q \ge 0$ and $\lim_{n \to \infty} \frac{q(\ln n)^{q-1}}{(p-r)n^{p-r}} = \lim_{n \to \infty} \frac{q}{(p-r)n^{p-r}(\ln n)^{1-q}} = 0$, otherwise, we apply L'Hopital's Rule again. $\lim_{n \to \infty} \frac{q(q-1)(\ln n)^{q-2}\left(\frac{1}{n}\right)}{(p-r)^2 n^{p-r-1}}$ $= \lim_{n \to \infty} \frac{q(q-1)(\ln n)^{q-2}}{(p-r)^2 n^{p-r}}$. If $q - 2 \le 0 \Rightarrow 2 - q \ge 0$ and $\lim_{n \to \infty} \frac{q(q-1)(\ln n)^{q-2}}{(p-r)^2 n^{p-r}} = \lim_{n \to \infty} \frac{q(q-1)}{(p-r)^2 n^{p-r}(\ln n)^{2-q}} = 0$; otherwise, we apply L'Hopital's Rule again. Since q is finite, there is a positive integer k such that $q - k \le 0 \Rightarrow k - q \ge 0$. Thus, after k applications of L'Hopital's Rule we obtain $\lim_{n \to \infty} \frac{q(q-1)\cdots(q-k+1)(\ln n)^{q-k}}{(p-r)^k n^{p-r}} = \lim_{n \to \infty} \frac{q(q-1)\cdots(q-k+1)}{(p-r)^k n^{p-r}(\ln n)^{k-q}} = 0$. Since the limit is 0 in every case, by Limit Comparison Test, the series $\sum_{n=1}^{\infty} \frac{(\ln n)^q}{n^p}$ converges.

62. Let $-\infty < q < \infty$ and $p \le 1$. If $q = 0$, then $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p} = \sum_{n=2}^{\infty} \frac{1}{n^p}$, which is a divergent p-series. If $q > 0$, compare with $\sum_{n=2}^{\infty} \frac{1}{n^p}$, which is a divergent p-series. Then $\lim_{n \to \infty} \frac{\frac{(\ln n)^q}{n^p}}{1/n^p} = \lim_{n \to \infty} (\ln n)^q = \infty$. If $q < 0 \Rightarrow -q > 0$, compare with $\sum_{n=2}^{\infty} \frac{1}{n^r}$, where $0 < p < r \le 1$. $\lim_{n \to \infty} \frac{\frac{(\ln n)^q}{n^p}}{1/n^r} = \lim_{n \to \infty} \frac{(\ln n)^q}{n^{p-r}} = \lim_{n \to \infty} \frac{n^{r-p}}{(\ln n)^{-q}}$ since $r - p > 0$. Apply L'Hopital's to obtain $\lim_{n \to \infty} \frac{(r-p)n^{r-p-1}}{(-q)(\ln n)^{-q-1}\left(\frac{1}{n}\right)} = \lim_{n \to \infty} \frac{(r-p)n^{r-p}}{(-q)(\ln n)^{-q-1}}$. If $-q - 1 \le 0 \Rightarrow q + 1 \ge 0$ and $\lim_{n \to \infty} \frac{(r-p)n^{r-p}(\ln n)^{q+1}}{(-q)} = \infty$, otherwise, we apply L'Hopital's Rule again to obtain $\lim_{n \to \infty} \frac{(r-p)^2 n^{r-p-1}}{(-q)(-q-1)(\ln n)^{-q-2}\left(\frac{1}{n}\right)} = \lim_{n \to \infty} \frac{(r-p)^2 n^{r-p}}{(-q)(-q-1)(\ln n)^{-q-2}}$. If $-q - 2 \le 0 \Rightarrow q + 2 \ge 0$ and $\lim_{n \to \infty} \frac{(r-p)^2 n^{r-p}}{(-q)(-q-1)(\ln n)^{-q-2}} = \lim_{n \to \infty} \frac{(r-p)^2 n^{r-p}(\ln n)^{q+2}}{(-q)(-q-1)} = \infty$, otherwise, we apply L'Hopital's Rule again. Since q is finite, there is a positive integer k such that $-q - k \le 0 \Rightarrow q + k \ge 0$. Thus, after k applications of L'Hopital's Rule we obtain $\lim_{n \to \infty} \frac{(r-p)^k n^{r-p}}{(-q)(-q-1)\cdots(-q-k+1)(\ln n)^{-q-k}} = \lim_{n \to \infty} \frac{(r-p)^k n^{r-p}(\ln n)^{q+k}}{(-q)(-q-1)\cdots(-q-k+1)} = \infty$.

Since the limit is ∞ if $q > 0$ or if $q < 0$ and $p < 1$, by Limit comparison test, the series $\sum_{n=1}^{\infty} \frac{(\ln n)^q}{n^{p-r}}$ diverges. Finally if $q < 0$ and $p = 1$ then $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p} = \sum_{n=2}^{\infty} \frac{(\ln n)^q}{n}$. Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is a divergent p-series. For $n \ge 3$, $\ln n \ge 1$ $\Rightarrow (\ln n)^q \ge 1 \Rightarrow \frac{(\ln n)^q}{n} \ge \frac{1}{n}$. Thus $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n}$ diverges by Comparison Test. Thus, if $-\infty < q < \infty$ and $p \le 1$, the series $\sum_{n=1}^{\infty} \frac{(\ln n)^q}{n^{p-r}}$ diverges.

63. Converges by Exercise 61 with $q = 3$ and $p = 4$.

64. Diverges by Exercise 62 with $q = \frac{1}{2}$ and $p = \frac{1}{2}$.

65. Converges by Exercise 61 with $q = 1000$ and $p = 1.001$.

66. Diverges by Exercise 62 with $q = \frac{1}{5}$ and $p = 0.99$.

67. Converges by Exercise 61 with $q = -3$ and $p = 1.1$.

68. Diverges by Exercise 62 with $q = -\frac{1}{2}$ and $p = \frac{1}{2}$.

69. Example CAS commands:

Maple:

```
a := n -> 1./n^3/sin(n)^2;
s := k -> sum( a(n), n=1..k );                                # (a)]
limit( s(k), k=infinity );
pts := [seq( [k,s(k)], k=1..100 )]:                            # (b)
plot( pts, style=point, title="#69(b) (Section 10.4)" );
pts := [seq( [k,s(k)], k=1..200 )]:                            # (c)
plot( pts, style=point, title="#69(c) (Section 10.4)" );
pts := [seq( [k,s(k)], k=1..400 )]:                            # (d)
plot( pts, style=point, title="#69(d) (Section 10.4)" );
evalf( 355/113 );
```

Mathematica:

```
Clear[a, n, s, k, p]
a[n_]:= 1 / ( n^3 Sin[n]^2 )
s[k_]= Sum[ a[n], {n, 1, k}]
points[p_]:= Table[{k, N[s[k]]}, {k, 1, p}]
points[100]
ListPlot[points[100]]
points[200]
ListPlot[points[200]
points[400]
ListPlot[points[400], PlotRange → All]
```

To investigate what is happening around k = 355, you could do the following.

```
N[355/113]
N[π − 355/113]
Sin[355]//N
a[355]//N
N[s[354]]
```

N[s[355]]
N[s[356]]

70. (a) Let $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series. By Example 5 in Section 10.2, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1. By Theorem 8, $S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n(n+1)}\right)$ also converges.

(b) Since $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1 (from Example 5 in Section 10.2), $S = 1 + \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n(n+1)}\right) = 1 + \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$

(c) The new series is comparible to $\sum_{n=1}^{\infty} \frac{1}{n^3}$, so it will converge faster because its terms $\to 0$ faster than the terms of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(d) The series $1 + \sum_{n=1}^{1000} \frac{1}{n^2(n+1)}$ gives a better approximation. Using Mathematica, $1 + \sum_{n=1}^{1000} \frac{1}{n^2(n+1)} = 1.644933568$, while $\sum_{n=1}^{1000000} \frac{1}{n^2} = 1.644933067$. Note that $\frac{\pi^2}{6} = 1.644934067$. The error is 4.99×10^{-7} compared with 1×10^{-6}.

10.5 THE RATIO AND ROOT TESTS

1. $\frac{2^n}{n!} > 0$ for all $n \ge 1$; $\lim_{n\to\infty} \left(\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right) = \lim_{n\to\infty} \left(\frac{2^n \cdot 2}{(n+1)\cdot n!} \cdot \frac{n!}{2^n}\right) = \lim_{n\to\infty} \left(\frac{2}{n+1}\right) = 0 < 1 \Rightarrow \sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges

2. $\frac{n+2}{3^n} > 0$ for all $n \ge 1$; $\lim_{n\to\infty} \left(\frac{\frac{(n+1)+2}{3^{n+1}}}{\frac{n+2}{3^n}}\right) = \lim_{n\to\infty} \left(\frac{n+3}{3^n \cdot 3} \cdot \frac{3^n}{n+2}\right) = \lim_{n\to\infty} \left(\frac{n+3}{3n+6}\right) = \lim_{n\to\infty} \left(\frac{1}{3}\right) = \frac{1}{3} < 1 \Rightarrow \sum_{n=1}^{\infty} \frac{n+2}{3^n}$ converges

3. $\frac{(n-1)!}{(n+1)^2} > 0$ for all $n \ge 1$; $\lim_{n\to\infty} \left(\frac{\frac{((n+1)-1)!}{((n+1)+1)^2}}{\frac{(n-1)!}{(n+1)^2}}\right) = \lim_{n\to\infty} \left(\frac{n\cdot(n-1)!}{(n+2)^2} \cdot \frac{(n+1)^2}{(n-1)!}\right) = \lim_{n\to\infty} \left(\frac{n^3+2n^2+n}{n^2+4n+4}\right) = \lim_{n\to\infty} \left(\frac{3n^2+4n+1}{2n+4}\right)$
$= \lim_{n\to\infty} \left(\frac{6n+4}{2}\right) = \infty > 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ diverges

4. $\frac{2^{n+1}}{n\cdot 3^{n-1}} > 0$ for all $n \ge 1$; $\lim_{n\to\infty} \left(\frac{\frac{2^{(n+1)+1}}{(n+1)\cdot 3^{(n+1)-1}}}{\frac{2^{n+1}}{n\cdot 3^{n-1}}}\right) = \lim_{n\to\infty} \left(\frac{2^{n+1}\cdot 2}{(n+1)\cdot 3^{n-1}\cdot 3} \cdot \frac{n\cdot 3^{n-1}}{2^{n+1}}\right) = \lim_{n\to\infty} \left(\frac{2n}{3n+3}\right) = \lim_{n\to\infty} \left(\frac{2}{3}\right) = \frac{2}{3} < 1$
$\Rightarrow \sum_{n=1}^{\infty} \frac{2^{n+1}}{n\cdot 3^{n-1}}$ converges

5. $\frac{n^4}{4^n} > 0$ for all $n \ge 1$; $\lim_{n\to\infty} \left(\frac{\frac{(n+1)^4}{4^{n+1}}}{\frac{n^4}{4^n}}\right) = \lim_{n\to\infty} \left(\frac{(n+1)^4}{4^n\cdot 4} \cdot \frac{4^n}{n^4}\right) = \lim_{n\to\infty} \left(\frac{n^4+4n^3+6n^2+4n+1}{4n^4}\right)$
$= \lim_{n\to\infty} \left(\frac{1}{4} + \frac{1}{n} + \frac{3}{2n^2} + \frac{1}{n^3} + \frac{1}{4n^4}\right) = \frac{1}{4} < 1 \Rightarrow \sum_{n=1}^{\infty} \frac{n^4}{4^n}$ converges

6. $\frac{3^{n+2}}{\ln n} > 0$ for all $n \ge 2$; $\lim_{n\to\infty} \left(\frac{\frac{3^{(n+1)+2}}{\ln(n+1)}}{\frac{3^{n+2}}{\ln n}}\right) = \lim_{n\to\infty} \left(\frac{3^{n+2}\cdot 3}{\ln(n+1)} \cdot \frac{\ln n}{3^{n+2}}\right) = \lim_{n\to\infty} \left(\frac{3\ln n}{\ln(n+1)}\right) = \lim_{n\to\infty} \left(\frac{\frac{3}{n}}{\frac{1}{n+1}}\right) = \lim_{n\to\infty} \left(\frac{3n+3}{n}\right)$
$= \lim_{n\to\infty} \left(\frac{3}{1}\right) = 3 > 1 \Rightarrow \sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$ diverges

7. $\frac{n^2(n+2)!}{n!\,3^{2n}} > 0$ for all $n \ge 1$; $\lim_{n\to\infty} \left(\frac{\frac{(n+1)^2((n+1)+2)!}{(n+1)!\,3^{2(n+1)}}}{\frac{n^2(n+2)!}{n!\,3^{2n}}}\right) = \lim_{n\to\infty} \left(\frac{(n+1)^2(n+3)(n+2)!}{(n+1)\cdot n!\,3^{2n}\cdot 3^2} \cdot \frac{n!\,3^{2n}}{n^2(n+2)!}\right) = \lim_{n\to\infty} \left(\frac{n^3+5n^2+7n+3}{9n^3+9n^2}\right)$
$= \lim_{n\to\infty} \left(\frac{3n^2+15n+7}{27n^2+18n}\right) = \lim_{n\to\infty} \left(\frac{6n+15}{54n+18}\right) = \lim_{n\to\infty} \left(\frac{6}{54}\right) = \frac{1}{9} < 1 \Rightarrow \sum_{n=1}^{\infty} \frac{n^2(n+2)!}{n!\,3^{2n}}$ converges

8. $\frac{n\cdot 5^n}{(2n+3)\ln(n+1)} > 0$ for all $n \ge 1$; $\lim\limits_{n\to\infty}\left(\frac{\frac{(n+1)\cdot 5^{n+1}}{(2(n+1)+3)\ln((n+1)+1)}}{\frac{n\cdot 5^n}{(2n+3)\ln(n+1)}}\right) = \lim\limits_{n\to\infty}\left(\frac{(n+1)\cdot 5^n\cdot 5}{(2n+5)\ln(n+2)}\cdot\frac{(2n+3)\ln(n+1)}{n\cdot 5^n}\right)$
$= \lim\limits_{n\to\infty}\left(\frac{5(n+1)\cdot(2n+3)}{n(2n+5)}\cdot\frac{\ln(n+1)}{\ln(n+2)}\right) = \lim\limits_{n\to\infty}\left(\frac{10n^2+25n+15}{2n^2+5n}\right)\cdot\lim\limits_{n\to\infty}\left(\frac{\ln(n+1)}{\ln(n+2)}\right) = \lim\limits_{n\to\infty}\left(\frac{20n+25}{4n+5}\right)\cdot\lim\limits_{n\to\infty}\left(\frac{\frac{1}{n+1}}{\frac{1}{n+2}}\right)$
$= \lim\limits_{n\to\infty}\left(\frac{20}{4}\right)\cdot\lim\limits_{n\to\infty}\left(\frac{n+2}{n+1}\right) = 5\cdot\lim\limits_{n\to\infty}\left(\frac{1}{1}\right) = 5\cdot 1 = 5 > 1 \Rightarrow \sum\limits_{n=2}^{\infty}\frac{n\cdot 5^n}{(2n+3)\ln(n+1)}$ diverges

9. $\frac{7}{(2n+5)^n} \ge 0$ for all $n \ge 1$; $\lim\limits_{n\to\infty}\sqrt[n]{\frac{7}{(2n+5)^n}} = \lim\limits_{n\to\infty}\left(\frac{\sqrt[n]{7}}{2n+5}\right) = 0 < 1 \Rightarrow \sum\limits_{n=1}^{\infty}\frac{7}{(2n+5)^n}$ converges

10. $\frac{4^n}{(3n)^n} \ge 0$ for all $n \ge 1$; $\lim\limits_{n\to\infty}\sqrt[n]{\frac{4^n}{(3n)^n}} = \lim\limits_{n\to\infty}\left(\frac{4}{3n}\right) = 0 < 1 \Rightarrow \sum\limits_{n=1}^{\infty}\frac{4^n}{(3n)^n}$ converges

11. $\left(\frac{4n+3}{3n-5}\right)^n \ge 0$ for all $n \ge 2$; $\lim\limits_{n\to\infty}\sqrt[n]{\left(\frac{4n+3}{3n-5}\right)^n} = \lim\limits_{n\to\infty}\left(\frac{4n+3}{3n-5}\right) = \lim\limits_{n\to\infty}\left(\frac{4}{3}\right) = \frac{4}{3} > 1 \Rightarrow \sum\limits_{n=1}^{\infty}\left(\frac{4n+3}{3n-5}\right)^n$ diverges

12. $\left[\ln\left(e^2+\frac{1}{n}\right)\right]^{n+1} \ge 0$ for all $n \ge 1$; $\lim\limits_{n\to\infty}\sqrt[n]{\left[\ln\left(e^2+\frac{1}{n}\right)\right]^{n+1}} = \lim\limits_{n\to\infty}\left[\ln\left(e^2+\frac{1}{n}\right)\right]^{1+1/n} = \ln(e^2) = 2 > 1$
$\Rightarrow \sum\limits_{n=1}^{\infty}\left[\ln\left(e^2+\frac{1}{n}\right)\right]^{n+1}$ diverges

13. $\frac{8}{\left(3+\frac{1}{n}\right)^{2n}} \ge 0$ for all $n \ge 1$; $\lim\limits_{n\to\infty}\sqrt[n]{\frac{8}{\left(3+\frac{1}{n}\right)^{2n}}} = \lim\limits_{n\to\infty}\left(\frac{\sqrt[n]{8}}{\left(3+\frac{1}{n}\right)^2}\right) = \frac{1}{9} < 1 \Rightarrow \sum\limits_{n=1}^{\infty}\frac{8}{\left(3+\frac{1}{n}\right)^{2n}}$ converges

14. $\left[\sin\left(\frac{1}{\sqrt{n}}\right)\right]^n \ge 0$ for all $n \ge 1$; $\lim\limits_{n\to\infty}\sqrt[n]{\left[\sin\left(\frac{1}{\sqrt{n}}\right)\right]^n} = \lim\limits_{n\to\infty}\sin\left(\frac{1}{\sqrt{n}}\right) = \sin(0) = 0 < 1 \Rightarrow \sum\limits_{n=1}^{\infty}\left[\sin\left(\frac{1}{\sqrt{n}}\right)\right]^n$ converges

15. $\left(1-\frac{1}{n}\right)^{n^2} \ge 0$ for all $n \ge 1$; $\lim\limits_{n\to\infty}\sqrt[n]{\left(1-\frac{1}{n}\right)^{n^2}} = \lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^{-1} < 1 \Rightarrow \sum\limits_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n^2}$ converges

16. $\frac{1}{n^{1+n}} \ge 0$ for all $n \ge 2$; $\lim\limits_{n\to\infty}\sqrt[n]{\frac{1}{n^{1+n}}} = \lim\limits_{n\to\infty}\left(\frac{\sqrt[n]{1}}{n^{1/n+1}}\right) = \lim\limits_{n\to\infty}\left(\frac{\sqrt[n]{1}}{n\sqrt[n]{n}}\right) = 0 < 1 \Rightarrow \sum\limits_{n=2}^{\infty}\frac{1}{n^{1+n}}$ converges

17. converges by the Ratio Test: $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\frac{\left[\frac{(n+1)^{\sqrt{2}}}{2^{n+1}}\right]}{\left[\frac{n^{\sqrt{2}}}{2^n}\right]} = \lim\limits_{n\to\infty}\frac{(n+1)^{\sqrt{2}}}{2^{n+1}}\cdot\frac{2^n}{n^{\sqrt{2}}} = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{\sqrt{2}}\left(\frac{1}{2}\right) = \frac{1}{2} < 1$

18. converges by the Ratio Test: $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\frac{\left(\frac{(n+1)^2}{e^{n+1}}\right)}{\left(\frac{n^2}{e^n}\right)} = \lim\limits_{n\to\infty}\frac{(n+1)^2}{e^{n+1}}\cdot\frac{e^n}{n^2} = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^2\left(\frac{1}{e}\right) = \frac{1}{e} < 1$

19. diverges by the Ratio Test: $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\frac{\left(\frac{(n+1)!}{e^{n+1}}\right)}{\left(\frac{n!}{e^n}\right)} = \lim\limits_{n\to\infty}\frac{(n+1)!}{e^{n+1}}\cdot\frac{e^n}{n!} = \lim\limits_{n\to\infty}\frac{n+1}{e} = \infty$

20. diverges by the Ratio Test: $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\frac{\left(\frac{(n+1)!}{10^{n+1}}\right)}{\left(\frac{n!}{10^n}\right)} = \lim\limits_{n\to\infty}\frac{(n+1)!}{10^{n+1}}\cdot\frac{10^n}{n!} = \lim\limits_{n\to\infty}\frac{n}{10} = \infty$

21. converges by the Ratio Test: $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\frac{\left(\frac{(n+1)^{10}}{10^{n+1}}\right)}{\left(\frac{n^{10}}{10^n}\right)} = \lim\limits_{n\to\infty}\frac{(n+1)^{10}}{10^{n+1}}\cdot\frac{10^n}{n^{10}} = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{10}\left(\frac{1}{10}\right) = \frac{1}{10} < 1$

22. diverges; $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\frac{n-2}{n}\right)^n = \lim\limits_{n\to\infty} \left(1+\frac{-2}{n}\right)^n = e^{-2} \neq 0$

23. converges by the Direct Comparison Test: $\frac{2+(-1)^n}{(1.25)^n} = \left(\frac{4}{5}\right)^n [2+(-1)^n] \le \left(\frac{4}{5}\right)^n(3)$ which is the n^{th} term of a convergent geometric series

24. converges; a geometric series with $|r| = \left|-\frac{2}{3}\right| < 1$

25. diverges; $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(1-\frac{3}{n}\right)^n = \lim\limits_{n\to\infty} \left(1+\frac{-3}{n}\right)^n = e^{-3} \approx 0.05 \neq 0$

26. diverges; $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(1-\frac{1}{3n}\right)^n = \lim\limits_{n\to\infty} \left(1+\frac{\left(-\frac{1}{3}\right)}{n}\right)^n = e^{-1/3} \approx 0.72 \neq 0$

27. converges by the Direct Comparison Test: $\frac{\ln n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$ for $n \ge 2$, the n^{th} term of a convergent p-series.

28. converges by the nth-Root Test: $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \sqrt[n]{\frac{(\ln n)^n}{n^n}} = \lim\limits_{n\to\infty} \frac{((\ln n)^n)^{1/n}}{(n^n)^{1/n}} = \lim\limits_{n\to\infty} \frac{\ln n}{n} = \lim\limits_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{1} = 0 < 1$

29. diverges by the Direct Comparison Test: $\frac{1}{n} - \frac{1}{n^2} = \frac{n-1}{n^2} > \frac{1}{2}\left(\frac{1}{n}\right)$ for $n > 2$ or by the Limit Comparison Test (part 1) with $\frac{1}{n}$.

30. converges by the nth-Root Test: $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \sqrt[n]{\left(\frac{1}{n}-\frac{1}{n^2}\right)^n} = \lim\limits_{n\to\infty} \left(\left(\frac{1}{n}-\frac{1}{n^2}\right)^n\right)^{1/n} = \lim\limits_{n\to\infty} \left(\frac{1}{n}-\frac{1}{n^2}\right) = 0 < 1$

31. diverges by the Direct Comparison Test: $\frac{\ln n}{n} > \frac{1}{n}$ for $n \ge 3$

32. converges by the Ratio Test: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{(n+1)\ln(n+1)}{2^{n+1}} \cdot \frac{2^n}{n\ln(n)} = \frac{1}{2} < 1$

33. converges by the Ratio Test: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{(n+2)(n+3)}{(n+1)!} \cdot \frac{n!}{(n+1)(n+2)} = 0 < 1$

34. converges by the Ratio Test: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{(n+1)^3}{e^{n+1}} \cdot \frac{e^n}{n^3} = \frac{1}{e} < 1$

35. converges by the Ratio Test: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{(n+4)!}{3!\,(n+1)!\,3^{n+1}} \cdot \frac{3!\,n!\,3^n}{(n+3)!} = \lim\limits_{n\to\infty} \frac{n+4}{3(n+1)} = \frac{1}{3} < 1$

36. converges by the Ratio Test: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{(n+1)2^{n+1}(n+2)!}{3^{n+1}(n+1)!} \cdot \frac{3^n n!}{n2^n(n+1)!} = \lim\limits_{n\to\infty} \left(\frac{n+1}{n}\right)\left(\frac{2}{3}\right)\left(\frac{n+2}{n+1}\right) = \frac{2}{3} < 1$

37. converges by the Ratio Test: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{(n+1)!}{(2n+3)!} \cdot \frac{(2n+1)!}{n!} = \lim\limits_{n\to\infty} \frac{n+1}{(2n+3)(2n+2)} = 0 < 1$

38. converges by the Ratio Test: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim\limits_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim\limits_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}$
$= \lim\limits_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$

39. converges by the Root Test: $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \sqrt[n]{\frac{n}{(\ln n)^n}} = \lim\limits_{n\to\infty} \frac{\sqrt[n]{n}}{\ln n} = \lim\limits_{n\to\infty} \frac{1}{\ln n} = 0 < 1$

40. converges by the Root Test: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{(\ln n)^{n/2}}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt{\ln n}} = \frac{\lim_{n\to\infty} \sqrt[n]{n}}{\lim_{n\to\infty} \sqrt{\ln n}} = 0 < 1$
$\left(\lim_{n\to\infty} \sqrt[n]{n} = 1\right)$

41. converges by the Direct Comparison Test: $\frac{n!\ \ln n}{n(n+2)!} = \frac{\ln n}{n(n+1)(n+2)} < \frac{n}{n(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$
which is the nth-term of a convergent p-series

42. diverges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^3\, 2^{n+1}} \cdot \frac{n^3 2^n}{3^n} = \lim_{n\to\infty} \frac{n^3}{(n+1)^3}\left(\frac{3}{2}\right) = \frac{3}{2} > 1$

43. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{[(n+1)!]^2}{[2(n+1)]!} \cdot \frac{(2n)!}{[n!]^2} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{4} < 1$

44. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+5)(2^{n+1}+3)}{3^{n+1}+2} \cdot \frac{3^n+2}{(2n+3)(2^n+3)} = \lim_{n\to\infty} \left[\frac{2n+5}{2n+3} \cdot \frac{2\cdot 6^n+4\cdot 2^n+3\cdot 3^n+6}{3\cdot 6^n+9\cdot 3^n+2\cdot 2^n+6}\right]$
$= \lim_{n\to\infty} \left[\frac{2n+5}{2n+3}\right] \cdot \lim_{n\to\infty} \left[\frac{2\cdot 6^n+4\cdot 2^n+3\cdot 3^n+6}{3\cdot 6^n+9\cdot 3^n+2\cdot 2^n+6}\right] = 1 \cdot \frac{2}{3} = \frac{2}{3} < 1$

45. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{1+\sin n}{n}\right) a_n}{a_n} = 0 < 1$

46. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{1+\tan^{-1} n}{n}\right) a_n}{a_n} = \lim_{n\to\infty} \frac{1+\tan^{-1} n}{n} = 0$ since the numerator approaches $1 + \frac{\pi}{2}$ while the denominator tends to ∞

47. diverges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{3n-1}{2n+5}\right) a_n}{a_n} = \lim_{n\to\infty} \frac{3n-1}{2n+5} = \frac{3}{2} > 1$

48. diverges; $a_{n+1} = \frac{n}{n+1}\, a_n \Rightarrow a_{n+1} = \left(\frac{n}{n+1}\right)\left(\frac{n-1}{n}\, a_{n-1}\right) \Rightarrow a_{n+1} = \left(\frac{n}{n+1}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\, a_{n-2}\right)$
$\Rightarrow a_{n+1} = \left(\frac{n}{n+1}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right)\cdots\left(\frac{1}{2}\right) a_1 \Rightarrow a_{n+1} = \frac{a_1}{n+1} \Rightarrow a_{n+1} = \frac{3}{n+1}$, which is a constant times the general term of the diverging harmonic series

49. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{2}{n}\right) a_n}{a_n} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1$

50. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{\sqrt[n]{n}}{2}\right) a_n}{a_n} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{n} = \frac{1}{2} < 1$

51. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{1+\ln n}{n}\right) a_n}{a_n} = \lim_{n\to\infty} \frac{1+\ln n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1$

52. $\frac{n+\ln n}{n+10} > 0$ and $a_1 = \frac{1}{2} \Rightarrow a_n > 0$; $\ln n > 10$ for $n > e^{10} \Rightarrow n + \ln n > n + 10 \Rightarrow \frac{n+\ln n}{n+10} > 1$
$\Rightarrow a_{n+1} = \frac{n+\ln n}{n+10}\, a_n > a_n$; thus $a_{n+1} > a_n \ge \frac{1}{2} \Rightarrow \lim_{n\to\infty} a_n \ne 0$, so the series diverges by the nth-Term Test

53. diverges by the nth-Term Test: $a_1 = \frac{1}{3}$, $a_2 = \sqrt[2]{\frac{1}{3}}$, $a_3 = \sqrt[3]{\sqrt[2]{\frac{1}{3}}} = \sqrt[6]{\frac{1}{3}}$, $a_4 = \sqrt[4]{\sqrt[3]{\sqrt[2]{\frac{1}{3}}}} = \sqrt[4!]{\frac{1}{3}}, \ldots,$
$a_n = \sqrt[n!]{\frac{1}{3}} \Rightarrow \lim_{n\to\infty} a_n = 1$ because $\left\{\sqrt[n!]{\frac{1}{3}}\right\}$ is a subsequence of $\left\{\sqrt[n]{\frac{1}{3}}\right\}$ whose limit is 1 by Table 8.1

54. converges by the Direct Comparison Test: $a_1 = \frac{1}{2}$, $a_2 = \left(\frac{1}{2}\right)^2$, $a_3 = \left(\left(\frac{1}{2}\right)^2\right)^3 = \left(\frac{1}{2}\right)^6$, $a_4 = \left(\left(\frac{1}{2}\right)^6\right)^4 = \left(\frac{1}{2}\right)^{24}, \ldots$
$\Rightarrow a_n = \left(\frac{1}{2}\right)^{n!} < \left(\frac{1}{2}\right)^n$ which is the nth-term of a convergent geometric series

55. converges by the Ratio Test: $\lim\limits_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n \to \infty} \frac{2^{n+1}(n+1)!\,(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{2^n n!\, n!} = \lim\limits_{n \to \infty} \frac{2(n+1)(n+1)}{(2n+2)(2n+1)}$
$= \lim\limits_{n \to \infty} \frac{n+1}{2n+1} = \frac{1}{2} < 1$

56. diverges by the Ratio Test: $\lim\limits_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n \to \infty} \frac{(3n+3)!}{(n+1)!\,(n+2)!\,(n+3)!} \cdot \frac{n!\,(n+1)!\,(n+2)!}{(3n)!}$
$= \lim\limits_{n \to \infty} \frac{(3n+3)(3+2)(3n+1)}{(n+1)(n+2)(n+3)} = \lim\limits_{n \to \infty} 3\left(\frac{3n+2}{n+2}\right)\left(\frac{3n+1}{n+3}\right) = 3 \cdot 3 \cdot 3 = 27 > 1$

57. diverges by the Root Test: $\lim\limits_{n \to \infty} \sqrt[n]{a_n} \equiv \lim\limits_{n \to \infty} \sqrt[n]{\frac{(n!)^n}{(n^n)^2}} = \lim\limits_{n \to \infty} \frac{n!}{n^2} = \infty > 1$

58. converges by the Root Test: $\lim\limits_{n \to \infty} \sqrt[n]{\frac{(n!)^n}{n^{n^2}}} = \lim\limits_{n \to \infty} \sqrt[n]{\frac{(n!)^n}{(n^n)^n}} = \lim\limits_{n \to \infty} \frac{n!}{n^n} = \lim\limits_{n \to \infty} \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\left(\frac{3}{n}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{n}{n}\right)$
$\le \lim\limits_{n \to \infty} \frac{1}{n} = 0 < 1$

59. converges by the Root Test: $\lim\limits_{n \to \infty} \sqrt[n]{a_n} = \lim\limits_{n \to \infty} \sqrt[n]{\frac{n^n}{2^{n^2}}} = \lim\limits_{n \to \infty} \frac{n}{2^n} = \lim\limits_{n \to \infty} \frac{1}{2^n \ln 2} = 0 < 1$

60. diverges by the Root Test: $\lim\limits_{n \to \infty} \sqrt[n]{a_n} = \lim\limits_{n \to \infty} \sqrt[n]{\frac{n^n}{(2^n)^2}} = \lim\limits_{n \to \infty} \frac{n}{4} = \infty > 1$

61. converges by the Ratio Test: $\lim\limits_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n \to \infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)(2n+1)}{4^{n+1}2^{n+1}(n+1)!} \cdot \frac{4^n\, 2^n\, n!}{1 \cdot 3 \cdot \cdots \cdot (2n-1)} = \lim\limits_{n \to \infty} \frac{2n+1}{(4 \cdot 2)(n+1)} = \frac{1}{4} < 1$

62. converges by the Ratio Test: $a_n = \frac{1 \cdot 3 \cdots (2n-1)}{(2 \cdot 4 \cdots 2n)\,(3^n+1)} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n-1)(2n)}{(2 \cdot 4 \cdots 2n)^2\,(3^n+1)} = \frac{(2n)!}{(2^n n!)^2\,(3^n+1)}$
$\Rightarrow \lim\limits_{n \to \infty} \frac{(2n+2)!}{[2^{n+1}(n+1)!]^2\,(3^{n+1}+1)} \cdot \frac{(2^n n!)^2\,(3^n+1)}{(2n)!} = \lim\limits_{n \to \infty} \frac{(2n+1)(2n+2)\,(3^n+1)}{2^2(n+1)^2\,(3^{n+1}+1)}$
$= \lim\limits_{n \to \infty} \left(\frac{4n^2+6n+2}{4n^2+8n+4}\right) \frac{(1+3^{-n})}{(3+3^{-n})} = 1 \cdot \frac{1}{3} = \frac{1}{3} < 1$

63. Ratio: $\lim\limits_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n \to \infty} \frac{1}{(n+1)^p} \cdot \frac{n^p}{1} = \lim\limits_{n \to \infty} \left(\frac{n}{n+1}\right)^p = 1^p = 1 \Rightarrow$ no conclusion
Root: $\lim\limits_{n \to \infty} \sqrt[n]{a_n} = \lim\limits_{n \to \infty} \sqrt[n]{\frac{1}{n^p}} = \lim\limits_{n \to \infty} \frac{1}{(\sqrt[n]{n})^p} = \frac{1}{(1)^p} = 1 \Rightarrow$ no conclusion

64. Ratio: $\lim\limits_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n \to \infty} \frac{1}{(\ln(n+1))^p} \cdot \frac{(\ln n)^p}{1} = \left[\lim\limits_{n \to \infty} \frac{\ln n}{\ln(n+1)}\right]^p = \left[\lim\limits_{n \to \infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n+1}\right)}\right]^p = \left(\lim\limits_{n \to \infty} \frac{n+1}{n}\right)^p$
$= (1)^p = 1 \Rightarrow$ no conclusion
Root: $\lim\limits_{n \to \infty} \sqrt[n]{a_n} = \lim\limits_{n \to \infty} \sqrt[n]{\frac{1}{(\ln n)^p}} = \frac{1}{\left(\lim\limits_{n \to \infty} (\ln n)^{1/n}\right)^p}$; let $f(n) = (\ln n)^{1/n}$, then $\ln f(n) = \frac{\ln(\ln n)}{n}$
$\Rightarrow \lim\limits_{n \to \infty} \ln f(n) = \lim\limits_{n \to \infty} \frac{\ln(\ln n)}{n} = \lim\limits_{n \to \infty} \frac{\left(\frac{1}{n \ln n}\right)}{1} = \lim\limits_{n \to \infty} \frac{1}{n \ln n} = 0 \Rightarrow \lim\limits_{n \to \infty} (\ln n)^{1/n}$
$= \lim\limits_{n \to \infty} e^{\ln f(n)} = e^0 = 1$; therefore $\lim\limits_{n \to \infty} \sqrt[n]{a_n} = \frac{1}{\left(\lim\limits_{n \to \infty} (\ln n)^{1/n}\right)^p} = \frac{1}{(1)^p} = 1 \Rightarrow$ no conclusion

65. $a_n \le \frac{n}{2^n}$ for every n and the series $\sum\limits_{n=1}^{\infty} \frac{n}{2^n}$ converges by the Ratio Test since $\lim\limits_{n \to \infty} \frac{(n+1)}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2} < 1$
$\Rightarrow \sum\limits_{n=1}^{\infty} a_n$ converges by the Direct Comparison Test

66. $\frac{2^{n^2}}{n!} > 0$ for all $n \geq 1$; $\lim\limits_{n\to\infty}\left(\frac{\frac{2^{(n+1)^2}}{(n+1)!}}{\frac{2^{n^2}}{n!}}\right) = \lim\limits_{n\to\infty}\left(\frac{2^{n^2+2n+1}}{(n+1)\cdot n!}\cdot\frac{n!}{2^{n^2}}\right) = \lim\limits_{n\to\infty}\left(\frac{2^{2n+1}}{n+1}\right) = \lim\limits_{n\to\infty}\left(\frac{2\cdot 4^n}{n+1}\right) = \lim\limits_{n\to\infty}\left(\frac{2\cdot 4^n \ln 4}{1}\right)$
$= \infty > 1 \Rightarrow \sum\limits_{n=1}^{\infty}\frac{2^{n^2}}{n!}$ diverges

10.6 ALTERNATING SERIES, ABSOLUTE AND CONDITIONAL CONVERGENCE

1. converges by the Alternating Convergence Test since: $u_n = \frac{1}{\sqrt{n}} > 0$ for all $n \geq 1$; $n \geq 1 \Rightarrow n+1 \geq n \Rightarrow \sqrt{n+1} \geq \sqrt{n}$
$\Rightarrow \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} \Rightarrow u_{n+1} \leq u_n$; $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}\frac{1}{\sqrt{n}} = 0.$

2. converges absolutely $\Rightarrow$ converges by the Alternating Convergence Test since $\sum\limits_{n=1}^{\infty}|a_n| = \sum\limits_{n=1}^{\infty}\frac{1}{n^{3/2}}$ which is a convergent p-series

3. converges $\Rightarrow$ converges by Alternating Series Test since: $u_n = \frac{1}{n3^n} > 0$ for all $n \geq 1$; $n \geq 1 \Rightarrow n+1 \geq n \Rightarrow 3^{n+1} \geq 3^n$
$\Rightarrow (n+1)3^{n+1} \geq n\,3^n \Rightarrow \frac{1}{(n+1)3^{n+1}} \leq \frac{1}{n\,3^n} \Rightarrow u_{n+1} \leq u_n$; $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}\frac{1}{n\,3^n} = 0.$

4. converges $\Rightarrow$ converges by Alternating Series Test since: $u_n = \frac{4}{(\ln n)^2} > 0$ for all $n \geq 2$; $n \geq 2 \Rightarrow n+1 \geq n$
$\Rightarrow \ln(n+1) \geq \ln n \Rightarrow (\ln(n+1))^2 \geq (\ln n)^2 \Rightarrow \frac{1}{(\ln(n+1))^2} \leq \frac{1}{(\ln n)^2} \Rightarrow \frac{4}{(\ln(n+1))^2} \leq \frac{4}{(\ln n)^2} \Rightarrow u_{n+1} \leq u_n$;
$\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}\frac{4}{(\ln n)^2} = 0.$

5. converges $\Rightarrow$ converges by Alternating Series Test since: $u_n = \frac{n}{n^2+1} > 0$ for all $n \geq 1$; $n \geq 1 \Rightarrow 2n^2 + 2n \geq n^2 + n + 1$
$\Rightarrow n^3 + 2n^2 + 2n \geq n^3 + n^2 + n + 1 \Rightarrow n(n^2 + 2n + 2) \geq n^3 + n^2 + n + 1 \Rightarrow n\left((n+1)^2 + 1\right) \geq (n^2+1)(n+1)$
$\Rightarrow \frac{n}{n^2+1} \geq \frac{n+1}{(n+1)^2+1} \Rightarrow u_{n+1} \leq u_n$; $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}\frac{n}{n^2+1} = 0.$

6. diverges $\Rightarrow$ diverges by n^{th} Term Test for Divergence since: $\lim\limits_{n\to\infty}\frac{n^2+5}{n^2+4} = 1 \Rightarrow \lim\limits_{n\to\infty}(-1)^{n+1}\frac{n^2+5}{n^2+4}$ = does not exist

7. diverges $\Rightarrow$ diverges by n^{th} Term Test for Divergence since: $\lim\limits_{n\to\infty}\frac{2^n}{n^2} = \infty \Rightarrow \lim\limits_{n\to\infty}(-1)^{n+1}\frac{2^n}{n^2}$ = does not exist

8. converges absolutely $\Rightarrow$ converges by the Absolute Convergence Test since $\sum\limits_{n=1}^{\infty}|a_n| = \sum\limits_{n=1}^{\infty}\frac{10^n}{(n+1)!}$, which converges by the Ratio Test, since $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\frac{10}{n+2} = 0 < 1$

9. diverges by the nth-Term Test since for $n > 10 \Rightarrow \frac{n}{10} > 1 \Rightarrow \lim\limits_{n\to\infty}\left(\frac{n}{10}\right)^n \neq 0 \Rightarrow \sum\limits_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^n$ diverges

10. converges by the Alternating Series Test because $f(x) = \ln x$ is an increasing function of x $\Rightarrow \frac{1}{\ln x}$ is decreasing
$\Rightarrow u_n \geq u_{n+1}$ for $n \geq 1$; also $u_n \geq 0$ for $n \geq 1$ and $\lim\limits_{n\to\infty}\frac{1}{\ln n} = 0$

11. converges by the Alternating Series Test since $f(x) = \frac{\ln x}{x} \Rightarrow f'(x) = \frac{1-\ln x}{x^2} < 0$ when $x > e \Rightarrow f(x)$ is decreasing
$\Rightarrow u_n \geq u_{n+1}$; also $u_n \geq 0$ for $n \geq 1$ and $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}\frac{\ln n}{n} = \lim\limits_{n\to\infty}\frac{\left(\frac{1}{n}\right)}{1} = 0$

12. converges by the Alternating Series Test since $f(x) = \ln\left(1 + x^{-1}\right) \Rightarrow f'(x) = \frac{-1}{x(x+1)} < 0$ for $x > 0 \Rightarrow f(x)$ is decreasing $\Rightarrow u_n \ge u_{n+1}$; also $u_n \ge 0$ for $n \ge 1$ and $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \ln\left(1 + \frac{1}{n}\right) = \ln\left(\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)\right) = \ln 1 = 0$

13. converges by the Alternating Series Test since $f(x) = \frac{\sqrt{x}+1}{x+1} \Rightarrow f'(x) = \frac{1-x-2\sqrt{x}}{2\sqrt{x}\,(x+1)^2} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow u_n \ge u_{n+1}$; also $u_n \ge 0$ for $n \ge 1$ and $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{\sqrt{n}+1}{n+1} = 0$

14. diverges by the nth-Term Test since $\lim\limits_{n\to\infty} \frac{3\sqrt{n+1}}{\sqrt{n}+1} = \lim\limits_{n\to\infty} \frac{3\sqrt{1+\frac{1}{n}}}{1+\left(\frac{1}{\sqrt{n}}\right)} = 3 \ne 0$

15. converges absolutely since $\sum\limits_{n=1}^{\infty} |a_n| = \sum\limits_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$ a convergent geometric series

16. converges absolutely by the Direct Comparison Test since $\left|\frac{(-1)^{n+1}(0.1)^n}{n}\right| = \frac{1}{(10)^n n} < \left(\frac{1}{10}\right)^n$ which is the nth term of a convergent geometric series

17. converges conditionally since $\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} > 0$ and $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \Rightarrow$ convergence; but $\sum\limits_{n=1}^{\infty} |a_n| = \sum\limits_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a divergent p-series

18. converges conditionally since $\frac{1}{1+\sqrt{n}} > \frac{1}{1+\sqrt{n+1}} > 0$ and $\lim\limits_{n\to\infty} \frac{1}{1+\sqrt{n}} = 0 \Rightarrow$ convergence; but $\sum\limits_{n=1}^{\infty} |a_n| = \sum\limits_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ is a divergent series since $\frac{1}{1+\sqrt{n}} \ge \frac{1}{2\sqrt{n}}$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a divergent p-series

19. converges absolutely since $\sum\limits_{n=1}^{\infty} |a_n| = \sum\limits_{n=1}^{\infty} \frac{n}{n^3+1}$ and $\frac{n}{n^3+1} < \frac{1}{n^2}$ which is the nth-term of a converging p-series

20. diverges by the nth-Term Test since $\lim\limits_{n\to\infty} \frac{n!}{2^n} = \infty$

21. converges conditionally since $\frac{1}{n+3} > \frac{1}{(n+1)+3} > 0$ and $\lim\limits_{n\to\infty} \frac{1}{n+3} = 0 \Rightarrow$ convergence; but $\sum\limits_{n=1}^{\infty} |a_n| = \sum\limits_{n=1}^{\infty} \frac{1}{n+3}$ diverges because $\frac{1}{n+3} \ge \frac{1}{4n}$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is a divergent series

22. converges absolutely because the series $\sum\limits_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|$ converges by the Direct Comparison Test since $\left|\frac{\sin n}{n^2}\right| \le \frac{1}{n^2}$

23. diverges by the nth-Term Test since $\lim\limits_{n\to\infty} \frac{3+n}{5+n} = 1 \ne 0$

24. converges absolutely by the Direct Comparison Test since $\left|\frac{(-2)^{n+1}}{n+5^n}\right| = \frac{2^{n+1}}{n+5^n} < 2\left(\frac{2}{5}\right)^n$ which is the nth term of a convergent geometric series

25. converges conditionally since $f(x) = \frac{1}{x^2} + \frac{1}{x} \Rightarrow f'(x) = -\left(\frac{2}{x^3} + \frac{1}{x^2}\right) < 0 \Rightarrow f(x)$ is decreasing and hence $u_n > u_{n+1} > 0$ for $n \ge 1$ and $\lim\limits_{n\to\infty} \left(\frac{1}{n^2} + \frac{1}{n}\right) = 0 \Rightarrow$ convergence; but $\sum\limits_{n=1}^{\infty} |a_n| = \sum\limits_{n=1}^{\infty} \frac{1+n}{n^2}$ $= \sum\limits_{n=1}^{\infty} \frac{1}{n^2} + \sum\limits_{n=1}^{\infty} \frac{1}{n}$ is the sum of a convergent and divergent series, and hence diverges

26. diverges by the nth-Term Test since $\lim_{n \to \infty} a_n = \lim_{n \to \infty} 10^{1/n} = 1 \neq 0$

27. converges absolutely by the Ratio Test: $\lim_{n \to \infty} \left(\frac{u_{n+1}}{u_n}\right) = \lim_{n \to \infty} \left[\frac{(n+1)^2 \left(\frac{2}{3}\right)^{n+1}}{n^2 \left(\frac{2}{3}\right)^n}\right] = \frac{2}{3} < 1$

28. converges conditionally since $f(x) = \frac{1}{x \ln x} \Rightarrow f'(x) = -\frac{[\ln(x)+1]}{(x \ln x)^2} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow u_n > u_{n+1} > 0$ for $n \geq 2$ and $\lim_{n \to \infty} \frac{1}{n \ln n} = 0 \Rightarrow$ convergence; but by the Integral Test,
$$\int_2^\infty \frac{dx}{x \ln x} = \lim_{b \to \infty} \int_2^b \left(\frac{\left(\frac{1}{x}\right)}{\ln x}\right) dx = \lim_{b \to \infty} [\ln(\ln x)]_2^b = \lim_{b \to \infty} [\ln(\ln b) - \ln(\ln 2)] = \infty$$
$\Rightarrow \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n \ln n}$ diverges

29. converges absolutely by the Integral Test since $\int_1^\infty (\tan^{-1} x)\left(\frac{1}{1+x^2}\right) dx = \lim_{b \to \infty} \left[\frac{(\tan^{-1} x)^2}{2}\right]_1^b$
$= \lim_{b \to \infty} \left[(\tan^{-1} b)^2 - (\tan^{-1} 1)^2\right] = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right] = \frac{3\pi^2}{32}$

30. converges conditionally since $f(x) = \frac{\ln x}{x - \ln x} \Rightarrow f'(x) = \frac{\left(\frac{1}{x}\right)(x - \ln x) - (\ln x)\left(1 - \frac{1}{x}\right)}{(x - \ln x)^2}$
$= \frac{1 - \left(\frac{\ln x}{x}\right) - \ln x + \left(\frac{\ln x}{x}\right)}{(x - \ln x)^2} = \frac{1 - \ln x}{(x - \ln x)^2} < 0 \Rightarrow u_n \geq u_{n+1} > 0$ when $n > e$ and $\lim_{n \to \infty} \frac{\ln n}{n - \ln n}$
$= \lim_{n \to \infty} \frac{\left(\frac{1}{n}\right)}{1 - \left(\frac{1}{n}\right)} = 0 \Rightarrow$ convergence; but $n - \ln n < n \Rightarrow \frac{1}{n - \ln n} > \frac{1}{n} \Rightarrow \frac{\ln n}{n - \ln n} > \frac{1}{n}$ so that
$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{\ln n}{n - \ln n}$ diverges by the Direct Comparison Test

31. diverges by the nth-Term Test since $\lim_{n \to \infty} \frac{n}{n+1} = 1 \neq 0$

32. converges absolutely since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$ is a convergent geometric series

33. converges absolutely by the Ratio Test: $\lim_{n \to \infty} \left(\frac{u_{n+1}}{u_n}\right) = \lim_{n \to \infty} \frac{(100)^{n+1}}{(n+1)!} \cdot \frac{n!}{(100)^n} = \lim_{n \to \infty} \frac{100}{n+1} = 0 < 1$

34. converges absolutely by the Direct Comparison Test since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2+2n+1}$ and $\frac{1}{n^2+2n+1} < \frac{1}{n^2}$ which is the nth-term of a convergent p-series

35. converges absolutely since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series

36. converges conditionally since $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is the convergent alternating harmonic series, but
$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

37. converges absolutely by the Root Test: $\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \left(\frac{(n+1)^n}{(2n)^n}\right)^{1/n} = \lim_{n \to \infty} \frac{n+1}{2n} = \frac{1}{2} < 1$

38. converges absolutely by the Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{((n+1)!)^2}{((2n+2)!)} \cdot \frac{(2n)!}{(n!)^2} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$

39. diverges by the nth-Term Test since $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{(2n)!}{2^n n!\, n} = \lim_{n\to\infty} \frac{(n+1)(n+2)\cdots(2n)}{2^n n}$
$= \lim_{n\to\infty} \frac{(n+1)(n+2)\cdots(n+(n-1))}{2^{n-1}} > \lim_{n\to\infty} \left(\frac{n+1}{2}\right)^{n-1} = \infty \neq 0$

40. converges absolutely by the Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!\,(n+1)!\,3^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{n!\,n!\,3^n}$
$= \lim_{n\to\infty} \frac{(n+1)^2\, 3}{(2n+2)(2n+3)} = \frac{3}{4} < 1$

41. converges conditionally since $\frac{\sqrt{n+1}-\sqrt{n}}{1} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ and $\left\{\frac{1}{\sqrt{n+1}+\sqrt{n}}\right\}$ is a decreasing sequence of positive terms which converges to 0 $\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$ converges; but $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges by the Limit Comparison Test (part 1) with $\frac{1}{\sqrt{n}}$; a divergent p-series:
$\lim_{n\to\infty} \left(\frac{\frac{1}{\sqrt{n+1}+\sqrt{n}}}{\frac{1}{\sqrt{n}}}\right) = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$

42. diverges by the nth-Term Test since $\lim_{n\to\infty} \left(\sqrt{n^2+n}-n\right) = \lim_{n\to\infty} \left(\sqrt{n^2+n}-n\right) \cdot \left(\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}\right)$
$= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2} \neq 0$

43. diverges by the nth-Term Test since $\lim_{n\to\infty} \left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right) = \lim_{n\to\infty} \left[\left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right)\left(\frac{\sqrt{n+\sqrt{n}}+\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}}\right)\right]$
$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{\sqrt{n}}}+1} = \frac{1}{2} \neq 0$

44. converges conditionally since $\left\{\frac{1}{\sqrt{n}+\sqrt{n+1}}\right\}$ is a decreasing sequence of positive terms converging to 0
$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ converges; but $\lim_{n\to\infty} \frac{\left(\frac{1}{\sqrt{n}+\sqrt{n+1}}\right)}{\left(\frac{1}{\sqrt{n}}\right)} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}} = \lim_{n\to\infty} \frac{1}{1+\sqrt{1+\frac{1}{n}}} = \frac{1}{2}$
so that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$ diverges by the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a divergent p-series

45. converges absolutely by the Direct Comparison Test since $\operatorname{sech}(n) = \frac{2}{e^n+e^{-n}} = \frac{2e^n}{e^{2n}+1} < \frac{2e^n}{e^{2n}} = \frac{2}{e^n}$ which is the nth term of a convergent geometric series

46. converges absolutely by the Limit Comparison Test (part 1): $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{2}{e^n-e^{-n}}$
Apply the Limit Comparison Test with $\frac{1}{e^n}$, the n-th term of a convergent geometric series:
$\lim_{n\to\infty} \left(\frac{\frac{2}{e^n-e^{-n}}}{\frac{1}{e^n}}\right) = \lim_{n\to\infty} \frac{2e^n}{e^n-e^{-n}} = \lim_{n\to\infty} \frac{2}{1-e^{-2n}} = 2$

47. $\frac{1}{4} - \frac{1}{6} + \frac{1}{8} - \frac{1}{10} + \frac{1}{12} - \frac{1}{14} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2(n+1)}$; converges by Alternating Series Test since: $u_n = \frac{1}{2(n+1)} > 0$ for all $n \geq 1$; $n+2 \geq n+1 \Rightarrow 2(n+2) \geq 2(n+1) \Rightarrow \frac{1}{2((n+1)+1)} \leq \frac{1}{2(n+1)} \Rightarrow u_{n+1} \leq u_n$; $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{2(n+1)} = 0.$

48. $1 + \frac{1}{4} - \frac{1}{9} - \frac{1}{16} + \frac{1}{25} + \frac{1}{36} - \frac{1}{49} - \frac{1}{64} + \ldots = \sum_{n=1}^{\infty} a_n$; converges by the Absolute Convergence Test since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series

49. $|\text{error}| < \left|(-1)^6 \left(\frac{1}{5}\right)\right| = 0.2$

50. $|\text{error}| < \left|(-1)^6 \left(\frac{1}{10^5}\right)\right| = 0.00001$

51. $|\text{error}| < \left|(-1)^6 \frac{(0.01)^5}{5}\right| = 2 \times 10^{-11}$

52. $|\text{error}| < |(-1)^4 t^4| = t^4 < 1$

53. $|\text{error}| < 0.001 \Rightarrow u_{n+1} < 0.001 \Rightarrow \frac{1}{(n+1)^2+3} < 0.001 \Rightarrow (n+1)^2 + 3 > 1000 \Rightarrow n > -1 + \sqrt{997} \approx 30.5753 \Rightarrow n \ge 31$

54. $|\text{error}| < 0.001 \Rightarrow u_{n+1} < 0.001 \Rightarrow \frac{n+1}{(n+1)^2+1} < 0.001 \Rightarrow (n+1)^2 + 1 > 1000(n+1) \Rightarrow n > \frac{998+\sqrt{998^2+4(998)}}{2}$
$\approx 998.9999 \Rightarrow n \ge 999$

55. $|\text{error}| < 0.001 \Rightarrow u_{n+1} < 0.001 \Rightarrow \frac{1}{\left((n+1)+3\sqrt{n+1}\right)^3} < 0.001 \Rightarrow \left((n+1) + 3\sqrt{n+1}\right)^3 > 1000$
$\Rightarrow \left(\sqrt{n+1}\right)^2 + 3\sqrt{n+1} - 10 > 0 \Rightarrow \sqrt{n+1} = -\frac{3+\sqrt{9+40}}{2} = 2 \Rightarrow n = 3 \Rightarrow n \ge 4$

56. $|\text{error}| < 0.001 \Rightarrow u_{n+1} < 0.001 \Rightarrow \frac{1}{\ln(\ln(n+3))} < 0.001 \Rightarrow \ln(\ln(n+3)) > 1000 \Rightarrow n > -3 + e^{e^{1000}} \approx 5.297 \times 10^{323228467}$ which is the maximum arbitrary-precision number represented by Mathematica on the particular computer solving this problem..

57. $\frac{1}{(2n)!} < \frac{5}{10^6} \Rightarrow (2n)! > \frac{10^6}{5} = 200{,}000 \Rightarrow n \ge 5 \Rightarrow 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} \approx 0.54030$

58. $\frac{1}{n!} < \frac{5}{10^6} \Rightarrow \frac{10^6}{5} < n! \Rightarrow n \ge 9 \Rightarrow 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} \approx 0.367881944$

59. (a) $a_n \ge a_{n+1}$ fails since $\frac{1}{3} < \frac{1}{2}$

(b) Since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left[\left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)^n\right] = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is the sum of two absolutely convergent series, we can rearrange the terms of the original series to find its sum:
$\left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right) = \frac{\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)} - \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)} = \frac{1}{2} - 1 = -\frac{1}{2}$

60. $s_{20} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{19} - \frac{1}{20} \approx 0.6687714032 \Rightarrow s_{20} + \frac{1}{2} \cdot \frac{1}{21} \approx 0.692580927$

61. The unused terms are $\sum_{j=n+1}^{\infty} (-1)^{j+1} a_j = (-1)^{n+1}(a_{n+1} - a_{n+2}) + (-1)^{n+3}(a_{n+3} - a_{n+4}) + \ldots$
$= (-1)^{n+1}\left[(a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + \ldots\right]$. Each grouped term is positive, so the remainder has the same sign as $(-1)^{n+1}$, which is the sign of the first unused term.

62. $s_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)} = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$
$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$ which are the first 2n terms of the first series, hence the two series are the same. Yes, for
$s_n = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$

$\Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 \Rightarrow$ both series converge to 1. The sum of the first $2n+1$ terms of the first series is $\left(1 - \frac{1}{n+1}\right) + \frac{1}{n+1} = 1$. Their sum is $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$.

63. Theorem 16 states that $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges. But this is equivalent to $\sum_{n=1}^{\infty} a_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} |a_n|$ diverges

64. $|a_1 + a_2 + \ldots + a_n| \le |a_1| + |a_2| + \ldots + |a_n|$ for all n; then $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges and these imply that $\left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty} |a_n|$

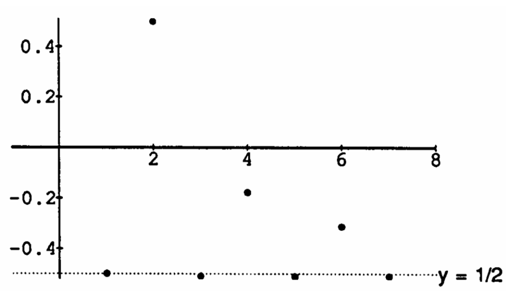
65. (a) $\sum_{n=1}^{\infty} |a_n + b_n|$ converges by the Direct Comparison Test since $|a_n + b_n| \le |a_n| + |b_n|$ and hence $\sum_{n=1}^{\infty} (a_n + b_n)$ converges absolutely

(b) $\sum_{n=1}^{\infty} |b_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} -b_n$ converges absolutely; since $\sum_{n=1}^{\infty} a_n$ converges absolutely and $\sum_{n=1}^{\infty} -b_n$ converges absolutely, we have $\sum_{n=1}^{\infty} [a_n + (-b_n)] = \sum_{n=1}^{\infty} (a_n - b_n)$ converges absolutely by part (a)

(c) $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow |k| \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} |ka_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} ka_n$ converges absolutely

66. If $a_n = b_n = (-1)^n \frac{1}{\sqrt{n}}$, then $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges, but $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

67. $s_1 = -\frac{1}{2}$, $s_2 = -\frac{1}{2} + 1 = \frac{1}{2}$,
$s_3 = -\frac{1}{2} + 1 - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} - \frac{1}{18} - \frac{1}{20} - \frac{1}{22} \approx -0.5099$,
$s_4 = s_3 + \frac{1}{3} \approx -0.1766$,
$s_5 = s_4 - \frac{1}{24} - \frac{1}{26} - \frac{1}{28} - \frac{1}{30} - \frac{1}{32} - \frac{1}{34} - \frac{1}{36} - \frac{1}{38} - \frac{1}{40} - \frac{1}{42} - \frac{1}{44} \approx -0.512$,
$s_6 = s_5 + \frac{1}{5} \approx -0.312$,
$s_7 = s_6 - \frac{1}{46} - \frac{1}{48} - \frac{1}{50} - \frac{1}{52} - \frac{1}{54} - \frac{1}{56} - \frac{1}{58} - \frac{1}{60} - \frac{1}{62} - \frac{1}{64} - \frac{1}{66} \approx -0.51106$



68. (a) Since $\sum |a_n|$ converges, say to M, for $\epsilon > 0$ there is an integer N_1 such that $\left|\sum_{n=1}^{N_1-1} |a_n| - M\right| < \frac{\epsilon}{2}$
$\Leftrightarrow \left|\sum_{n=1}^{N_1-1} |a_n| - \left(\sum_{n=1}^{N_1-1} |a_n| + \sum_{n=N_1}^{\infty} |a_n|\right)\right| < \frac{\epsilon}{2} \Leftrightarrow \left|-\sum_{n=N_1}^{\infty} |a_n|\right| < \frac{\epsilon}{2} \Leftrightarrow \sum_{n=N_1}^{\infty} |a_n| < \frac{\epsilon}{2}$. Also, $\sum a_n$ converges to L $\Leftrightarrow$ for $\epsilon > 0$ there is an integer N_2 (which we can choose greater than or equal to N_1) such that $|s_{N_2} - L| < \frac{\epsilon}{2}$. Therefore, $\sum_{n=N_1}^{\infty} |a_n| < \frac{\epsilon}{2}$ and $|s_{N_2} - L| < \frac{\epsilon}{2}$.

(b) The series $\sum_{n=1}^{\infty} |a_n|$ converges absolutely, say to M. Thus, there exists N_1 such that $\left|\sum_{n=1}^{k} |a_n| - M\right| < \epsilon$ whenever $k > N_1$. Now all of the terms in the sequence $\{|b_n|\}$ appear in $\{|a_n|\}$. Sum together all of the terms in $\{|b_n|\}$, in order, until you include all of the terms $\{|a_n|\}_{n=1}^{N_1}$, and let N_2 be the largest index in the sum $\sum_{n=1}^{N_2} |b_n|$ so obtained. Then $\left|\sum_{n=1}^{N_2} |b_n| - M\right| < \epsilon$ as well $\Rightarrow \sum_{n=1}^{\infty} |b_n|$ converges to M.

10.7 POWER SERIES

1. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\right| < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} (-1)^n$, a divergent series; when $x = 1$ we have $\sum_{n=1}^{\infty} 1$, a divergent series
   (a) the radius is 1; the interval of convergence is $-1 < x < 1$
   (b) the interval of absolute convergence is $-1 < x < 1$
   (c) there are no values for which the series converges conditionally

2. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x+5)^{n+1}}{(x+5)^n}\right| < 1 \Rightarrow |x+5| < 1 \Rightarrow -6 < x < -4$; when $x = -6$ we have $\sum_{n=1}^{\infty} (-1)^n$, a divergent series; when $x = -4$ we have $\sum_{n=1}^{\infty} 1$, a divergent series
   (a) the radius is 1; the interval of convergence is $-6 < x < -4$
   (b) the interval of absolute convergence is $-6 < x < -4$
   (c) there are no values for which the series converges conditionally

3. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(4x+1)^{n+1}}{(4x+1)^n}\right| < 1 \Rightarrow |4x+1| < 1 \Rightarrow -1 < 4x+1 < 1 \Rightarrow -\frac{1}{2} < x < 0$; when $x = -\frac{1}{2}$ we have $\sum_{n=1}^{\infty} (-1)^n(-1)^n = \sum_{n=1}^{\infty} (-1)^{2n} = \sum_{n=1}^{\infty} 1^n$, a divergent series; when $x = 0$ we have $\sum_{n=1}^{\infty} (-1)^n(1)^n = \sum_{n=1}^{\infty} (-1)^n$, a divergent series
   (a) the radius is $\frac{1}{4}$; the interval of convergence is $-\frac{1}{2} < x < 0$
   (b) the interval of absolute convergence is $-\frac{1}{2} < x < 0$
   (c) there are no values for which the series converges conditionally

4. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(3x-2)^{n+1}}{n+1} \cdot \frac{n}{(3x-2)^n}\right| < 1 \Rightarrow |3x-2| \lim_{n\to\infty} \left(\frac{n}{n+1}\right) < 1 \Rightarrow |3x-2| < 1$ $\Rightarrow -1 < 3x-2 < 1 \Rightarrow \frac{1}{3} < x < 1$; when $x = \frac{1}{3}$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is the alternating harmonic series and is conditionally convergent; when $x = 1$ we have $\sum_{n=1}^{\infty} \frac{1}{n}$, the divergent harmonic series
   (a) the radius is $\frac{1}{3}$; the interval of convergence is $\frac{1}{3} \le x < 1$
   (b) the interval of absolute convergence is $\frac{1}{3} < x < 1$
   (c) the series converges conditionally at $x = \frac{1}{3}$

5. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x-2)^{n+1}}{10^{n+1}} \cdot \frac{10^n}{(x-2)^n}\right| < 1 \Rightarrow \frac{|x-2|}{10} < 1 \Rightarrow |x-2| < 10 \Rightarrow -10 < x-2 < 10$ $\Rightarrow -8 < x < 12$; when $x = -8$ we have $\sum_{n=1}^{\infty} (-1)^n$, a divergent series; when $x = 12$ we have $\sum_{n=1}^{\infty} 1$, a divergent series
   (a) the radius is 10; the interval of convergence is $-8 < x < 12$
   (b) the interval of absolute convergence is $-8 < x < 12$
   (c) there are no values for which the series converges conditionally

6. $\lim\limits_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n \to \infty} \left|\frac{(2x)^{n+1}}{(2x)^n}\right| < 1 \Rightarrow \lim\limits_{n \to \infty} |2x| < 1 \Rightarrow |2x| < 1 \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$; when $x = -\frac{1}{2}$ we have $\sum\limits_{n=1}^{\infty} (-1)^n$, a divergent series; when $x = \frac{1}{2}$ we have $\sum\limits_{n=1}^{\infty} 1$, a divergent series
   (a) the radius is $\frac{1}{2}$; the interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$
   (b) the interval of absolute convergence is $-\frac{1}{2} < x < \frac{1}{2}$
   (c) there are no values for which the series converges conditionally

7. $\lim\limits_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n \to \infty} \left|\frac{(n+1)x^{n+1}}{(n+3)} \cdot \frac{(n+2)}{nx^n}\right| < 1 \Rightarrow |x| \lim\limits_{n \to \infty} \frac{(n+1)(n+2)}{(n+3)(n)} < 1 \Rightarrow |x| < 1$
   $\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum\limits_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$, a divergent series by the nth-term Test; when $x = 1$ we have $\sum\limits_{n=1}^{\infty} \frac{n}{n+2}$, a divergent series
   (a) the radius is 1; the interval of convergence is $-1 < x < 1$
   (b) the interval of absolute convergence is $-1 < x < 1$
   (c) there are no values for which the series converges conditionally

8. $\lim\limits_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n \to \infty} \left|\frac{(x+2)^{n+1}}{n+1} \cdot \frac{n}{(x+2)^n}\right| < 1 \Rightarrow |x+2| \lim\limits_{n \to \infty} \left(\frac{n}{n+1}\right) < 1 \Rightarrow |x+2| < 1$
   $\Rightarrow -1 < x+2 < 1 \Rightarrow -3 < x < -1$; when $x = -3$ we have $\sum\limits_{n=1}^{\infty} \frac{1}{n}$, a divergent series; when $x = -1$ we have $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$, a convergent series
   (a) the radius is 1; the interval of convergence is $-3 < x \le -1$
   (b) the interval of absolute convergence is $-3 < x < -1$
   (c) the series converges conditionally at $x = -1$

9. $\lim\limits_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n \to \infty} \left|\frac{x^{n+1}}{(n+1)\sqrt{n+1}\,3^{n+1}} \cdot \frac{n\sqrt{n}\,3^n}{x^n}\right| < 1 \Rightarrow \frac{|x|}{3}\left(\lim\limits_{n \to \infty} \frac{n}{n+1}\right)\left(\sqrt{\lim\limits_{n \to \infty} \frac{n}{n+1}}\right) < 1$
   $\Rightarrow \frac{|x|}{3}(1)(1) < 1 \Rightarrow |x| < 3 \Rightarrow -3 < x < 3$; when $x = -3$ we have $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$, an absolutely convergent series; when $x = 3$ we have $\sum\limits_{n=1}^{\infty} \frac{1}{n^{3/2}}$, a convergent p-series
   (a) the radius is 3; the interval of convergence is $-3 \le x \le 3$
   (b) the interval of absolute convergence is $-3 \le x \le 3$
   (c) there are no values for which the series converges conditionally

10. $\lim\limits_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n \to \infty} \left|\frac{(x-1)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(x-1)^n}\right| < 1 \Rightarrow |x-1| \sqrt{\lim\limits_{n \to \infty} \frac{n}{n+1}} < 1 \Rightarrow |x-1| < 1$
   $\Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2$; when $x = 0$ we have $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$, a conditionally convergent series; when $x = 2$ we have $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1/2}}$, a divergent series
   (a) the radius is 1; the interval of convergence is $0 \le x < 2$
   (b) the interval of absolute convergence is $0 < x < 2$
   (c) the series converges conditionally at $x = 0$

11. $\lim\limits_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim\limits_{n \to \infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| < 1 \Rightarrow |x| \lim\limits_{n \to \infty} \left(\frac{1}{n+1}\right) < 1$ for all x
   (a) the radius is ∞; the series converges for all x

(b) the series converges absolutely for all x
(c) there are no values for which the series converges conditionally

12. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{3^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n x^n}\right| < 1 \Rightarrow 3|x| \lim_{n\to\infty} \left(\frac{1}{n+1}\right) < 1$ for all x
(a) the radius is ∞; the series converges for all x
(b) the series converges absolutely for all x
(c) there are no values for which the series converges conditionally

13. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{4^{n+1} x^{2n+2}}{n+1} \cdot \frac{n}{4^n x^{2n}}\right| < 1 \Rightarrow x^2 \lim_{n\to\infty} \left(\frac{4n}{n+1}\right) = 4x^2 < 1 \Rightarrow x^2 < \frac{1}{4}$
$\Rightarrow -\frac{1}{2} < x < \frac{1}{2}$; when $x = -\frac{1}{2}$ we have $\sum_{n=1}^{\infty} \frac{4^n}{n}\left(-\frac{1}{2}\right)^{2n} = \sum_{n=1}^{\infty} \frac{1}{n}$, a divergent p-series; when $x = \frac{1}{2}$ we have $\sum_{n=1}^{\infty} \frac{4^n}{n}\left(\frac{1}{2}\right)^{2n} = \sum_{n=1}^{\infty} \frac{1}{n}$, a divergent p-series
(a) the radius is $\frac{1}{2}$; the interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$
(b) the interval of absolute convergence is $-\frac{1}{2} < x < \frac{1}{2}$
(c) there are no values for which the series converges conditionally

14. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x-1)^{n+1}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{(x-1)^n}\right| < 1 \Rightarrow |x-1| \lim_{n\to\infty} \left(\frac{n^2}{3(n+1)^2}\right) = \frac{1}{3}|x-1| < 1$
$\Rightarrow -2 < x < 4$; when $x = -2$ we have $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, an absolutely convergent series; when $x = 4$ we have $\sum_{n=1}^{\infty} \frac{(3)^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, an absolutely convergent series.
(a) the radius is 3; the interval of convergence is $-2 \le x \le 4$
(b) the interval of absolute convergence is $-2 \le x \le 4$
(c) there are no values for which the series converges conditionally

15. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{\sqrt{(n+1)^2+3}} \cdot \frac{\sqrt{n^2+3}}{x^n}\right| < 1 \Rightarrow |x| \sqrt{\lim_{n\to\infty} \frac{n^2+3}{n^2+2n+4}} < 1 \Rightarrow |x| < 1$
$\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$, a conditionally convergent series; when $x = 1$ we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$, a divergent series
(a) the radius is 1; the interval of convergence is $-1 \le x < 1$
(b) the interval of absolute convergence is $-1 < x < 1$
(c) the series converges conditionally at $x = -1$

16. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{\sqrt{(n+1)^2+3}} \cdot \frac{\sqrt{n^2+3}}{x^n}\right| < 1 \Rightarrow |x| \sqrt{\lim_{n\to\infty} \frac{n^2+3}{n^2+2n+4}} < 1 \Rightarrow |x| < 1$
$\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$, a divergent series; when $x = 1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$, a conditionally convergent series
(a) the radius is 1; the interval of convergence is $-1 < x \le 1$
(b) the interval of absolute convergence is $-1 < x < 1$
(c) the series converges conditionally at $x = 1$

17. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(n+1)(x+3)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n(x+3)^n}\right| < 1 \Rightarrow \frac{|x+3|}{5} \lim_{n\to\infty} \left(\frac{n+1}{n}\right) < 1 \Rightarrow \frac{|x+3|}{5} < 1$

$\Rightarrow |x+3| < 5 \Rightarrow -5 < x+3 < 5 \Rightarrow -8 < x < 2$; when $x = -8$ we have $\sum_{n=1}^{\infty} \frac{n(-5)^n}{5^n} = \sum_{n=1}^{\infty} (-1)^n n$, a divergent series; when $x = 2$ we have $\sum_{n=1}^{\infty} \frac{n5^n}{5^n} = \sum_{n=1}^{\infty} n$, a divergent series

(a) the radius is 5; the interval of convergence is $-8 < x < 2$
(b) the interval of absolute convergence is $-8 < x < 2$
(c) there are no values for which the series converges conditionally

18. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{4^{n+1}(n^2+2n+2)} \cdot \frac{4^n(n^2+1)}{nx^n}\right| < 1 \Rightarrow \frac{|x|}{4} \lim_{n\to\infty} \left|\frac{(n+1)(n^2+1)}{n(n^2+2n+2)}\right| < 1 \Rightarrow |x| < 4$

$\Rightarrow -4 < x < 4$; when $x = -4$ we have $\sum_{n=1}^{\infty} \frac{n(-1)^n}{n^2+1}$, a conditionally convergent series; when $x = 4$ we have $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$, a divergent series

(a) the radius is 4; the interval of convergence is $-4 \le x < 4$
(b) the interval of absolute convergence is $-4 < x < 4$
(c) the series converges conditionally at $x = -4$

19. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{\sqrt{n+1}\,x^{n+1}}{3^{n+1}} \cdot \frac{3^n}{\sqrt{n}\,x^n}\right| < 1 \Rightarrow \frac{|x|}{3} \sqrt{\lim_{n\to\infty} \left(\frac{n+1}{n}\right)} < 1 \Rightarrow \frac{|x|}{3} < 1 \Rightarrow |x| < 3$

$\Rightarrow -3 < x < 3$; when $x = -3$ we have $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}$, a divergent series; when $x = 3$ we have $\sum_{n=1}^{\infty} \sqrt{n}$, a divergent series

(a) the radius is 3; the interval of convergence is $-3 < x < 3$
(b) the interval of absolute convergence is $-3 < x < 3$
(c) there are no values for which the series converges conditionally

20. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{\sqrt[n+1]{n+1}\,(2x+5)^{n+1}}{\sqrt[n]{n}\,(2x+5)^n}\right| < 1 \Rightarrow |2x+5| \lim_{n\to\infty} \left(\frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}}\right) < 1$

$\Rightarrow |2x+5| \left(\frac{\lim_{t\to\infty} \sqrt[t]{t}}{\lim_{n\to\infty} \sqrt[n]{n}}\right) < 1 \Rightarrow |2x+5| < 1 \Rightarrow -1 < 2x+5 < 1 \Rightarrow -3 < x < -2$; when $x = -3$ we have $\sum_{n=1}^{\infty} (-1)\sqrt[n]{n}$, a divergent series since $\lim_{n\to\infty} \sqrt[n]{n} = 1$; when $x = -2$ we have $\sum_{n=1}^{\infty} \sqrt[n]{n}$, a divergent series

(a) the radius is $\frac{1}{2}$; the interval of convergence is $-3 < x < -2$
(b) the interval of absolute convergence is $-3 < x < -2$
(c) there are no values for which the series converges conditionally

21. First, rewrite the series as $\sum_{n=1}^{\infty} (2+(-1)^n)(x+1)^{n-1} = \sum_{n=1}^{\infty} 2(x+1)^{n-1} + \sum_{n=1}^{\infty} (-1)^n (x+1)^{n-1}$. For the series $\sum_{n=1}^{\infty} 2(x+1)^{n-1}$: $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{2(x+1)^n}{2(x+1)^{n-1}}\right| < 1 \Rightarrow |x+1| \lim_{n\to\infty} 1 = |x+1| < 1 \Rightarrow -2 < x < 0$; For the series $\sum_{n=1}^{\infty} (-1)^n (x+1)^{n-1}$: $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(x+1)^n}{(-1)^n (x+1)^{n-1}}\right| < 1 \Rightarrow |x+1| \lim_{n\to\infty} 1 = |x+1| < 1$

$\Rightarrow -2 < x < 0$; when $x = -2$ we have $\sum_{n=1}^{\infty} (2+(-1)^n)(-1)^{n-1}$, a divergent series; when $x = 0$ we have $\sum_{n=1}^{\infty} (2+(-1)^n)$, a divergent series

(a) the radius is 1; the interval of convergence is $-2 < x < 0$
(b) the interval of absolute convergence is $-2 < x < 0$
(c) there are no values for which the series converges conditionally

22. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(-1)^{n+1}3^{2n+2}(x-2)^{n+1}}{3(n+1)}\cdot\frac{3n}{(-1)^n 3^{2n}(x-2)^n}\right| < 1 \Rightarrow |x-2|\lim_{n\to\infty}\frac{9n}{n+1} = 9|x-2| < 1$

$\Rightarrow \frac{17}{9} < x < \frac{19}{9}$; when $x = \frac{17}{9}$ we have $\sum_{n=1}^{\infty}\frac{(-1)^n 3^{2n}}{3n}\left(-\frac{1}{9}\right)^n = \sum_{n=1}^{\infty}\frac{1}{3n}$, a divergent series; when $x = \frac{19}{9}$ we have $\sum_{n=1}^{\infty}\frac{(-1)^n 3^{2n}}{3n}\left(\frac{1}{9}\right)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{3n}$, a conditionally convergent series.

(a) the radius is $\frac{1}{9}$; the interval of convergence is $\frac{17}{9} < x \le \frac{19}{9}$

(b) the interval of absolute convergence is $\frac{17}{9} < x < \frac{19}{9}$

(c) the series converges conditionally at $x = \frac{19}{9}$

23. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{\left(1+\frac{1}{n+1}\right)^{n+1}x^{n+1}}{\left(1+\frac{1}{n}\right)^n x^n}\right| < 1 \Rightarrow |x|\left(\frac{\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}\right) < 1 \Rightarrow |x|\left(\frac{e}{e}\right) < 1 \Rightarrow |x| < 1$

$\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty}(-1)^n\left(1+\frac{1}{n}\right)^n$, a divergent series by the nth-Term Test since $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \neq 0$; when $x = 1$ we have $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$, a divergent series

(a) the radius is 1; the interval of convergence is $-1 < x < 1$

(b) the interval of absolute convergence is $-1 < x < 1$

(c) there are no values for which the series converges conditionally

24. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{\ln(n+1)x^{n+1}}{x^n \ln n}\right| < 1 \Rightarrow |x|\lim_{n\to\infty}\left|\frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)}\right| < 1 \Rightarrow |x|\lim_{n\to\infty}\left(\frac{n}{n+1}\right) < 1 \Rightarrow |x| < 1$

$\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty}(-1)^n \ln n$, a divergent series by the nth-Term Test since $\lim_{n\to\infty}\ln n \neq 0$; when $x = 1$ we have $\sum_{n=1}^{\infty}\ln n$, a divergent series

(a) the radius is 1; the interval of convergence is $-1 < x < 1$

(b) the interval of absolute convergence is $-1 < x < 1$

(c) there are no values for which the series converges conditionally

25. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}x^{n+1}}{n^n x^n}\right| < 1 \Rightarrow |x|\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)\left(\lim_{n\to\infty}(n+1)\right) < 1$

$\Rightarrow e|x|\lim_{n\to\infty}(n+1) < 1 \Rightarrow$ only $x = 0$ satisfies this inequality

(a) the radius is 0; the series converges only for $x = 0$

(b) the series converges absolutely only for $x = 0$

(c) there are no values for which the series converges conditionally

26. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(n+1)!\,(x-4)^{n+1}}{n!\,(x-4)^n}\right| < 1 \Rightarrow |x-4|\lim_{n\to\infty}(n+1) < 1 \Rightarrow$ only $x = 4$ satisfies this inequality

(a) the radius is 0; the series converges only for $x = 4$

(b) the series converges absolutely only for $x = 4$

(c) there are no values for which the series converges conditionally

27. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(n+1)\,2^{n+1}}\cdot\frac{n2^n}{(x+2)^n}\right| < 1 \Rightarrow \frac{|x+2|}{2}\lim_{n\to\infty}\left(\frac{n}{n+1}\right) < 1 \Rightarrow \frac{|x+2|}{2} < 1 \Rightarrow |x+2| < 2$

$\Rightarrow -2 < x+2 < 2 \Rightarrow -4 < x < 0$; when $x = -4$ we have $\sum_{n=1}^{\infty}\frac{-1}{n}$, a divergent series; when $x = 0$ we have $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$, the alternating harmonic series which converges conditionally

(a) the radius is 2; the interval of convergence is $-4 < x \le 0$

(b) the interval of absolute convergence is $-4 < x < 0$

(c) the series converges conditionally at $x = 0$

28. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(n+2)(x-1)^{n+1}}{(-2)^n(n+1)(x-1)^n}\right| < 1 \Rightarrow 2\,|x-1| \lim_{n\to\infty}\left(\frac{n+2}{n+1}\right) < 1 \Rightarrow 2\,|x-1| < 1$
$\Rightarrow |x-1| < \frac{1}{2} \Rightarrow -\frac{1}{2} < x-1 < \frac{1}{2} \Rightarrow \frac{1}{2} < x < \frac{3}{2}$; when $x = \frac{1}{2}$ we have $\sum_{n=1}^{\infty}(n+1)$, a divergent series; when $x = \frac{3}{2}$ we have $\sum_{n=1}^{\infty}(-1)^n(n+1)$, a divergent series

(a) the radius is $\frac{1}{2}$; the interval of convergence is $\frac{1}{2} < x < \frac{3}{2}$
(b) the interval of absolute convergence is $\frac{1}{2} < x < \frac{3}{2}$
(c) there are no values for which the series converges conditionally

29. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)\,(\ln(n+1))^2}\cdot\frac{n(\ln n)^2}{x^n}\right| < 1 \Rightarrow |x|\left(\lim_{n\to\infty}\frac{n}{n+1}\right)\left(\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}\right)^2 < 1$
$\Rightarrow |x|\,(1)\left(\lim_{n\to\infty}\frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n+1}\right)}\right)^2 < 1 \Rightarrow |x|\left(\lim_{n\to\infty}\frac{n+1}{n}\right)^2 < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty}\frac{(-1)^n}{n(\ln n)^2}$ which converges absolutely; when $x = 1$ we have $\sum_{n=1}^{\infty}\frac{1}{n(\ln n)^2}$ which converges

(a) the radius is 1; the interval of convergence is $-1 \le x \le 1$
(b) the interval of absolute convergence is $-1 \le x \le 1$
(c) there are no values for which the series converges conditionally

30. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)\ln(n+1)}\cdot\frac{n\ln(n)}{x^n}\right| < 1 \Rightarrow |x|\left(\lim_{n\to\infty}\frac{n}{n+1}\right)\left(\lim_{n\to\infty}\frac{\ln(n)}{\ln(n+1)}\right) < 1$
$\Rightarrow |x|\,(1)(1) < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$, a convergent alternating series; when $x = 1$ we have $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ which diverges by Exercise 38, Section 9.3

(a) the radius is 1; the interval of convergence is $-1 \le x < 1$
(b) the interval of absolute convergence is $-1 < x < 1$
(c) the series converges conditionally at $x = -1$

31. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(4x-5)^{2n+3}}{(n+1)^{3/2}}\cdot\frac{n^{3/2}}{(4x-5)^{2n+1}}\right| < 1 \Rightarrow (4x-5)^2\left(\lim_{n\to\infty}\frac{n}{n+1}\right)^{3/2} < 1 \Rightarrow (4x-5)^2 < 1$
$\Rightarrow |4x-5| < 1 \Rightarrow -1 < 4x-5 < 1 \Rightarrow 1 < x < \frac{3}{2}$; when $x = 1$ we have $\sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n^{3/2}} = \sum_{n=1}^{\infty}\frac{-1}{n^{3/2}}$ which is absolutely convergent; when $x = \frac{3}{2}$ we have $\sum_{n=1}^{\infty}\frac{(1)^{2n+1}}{n^{3/2}}$, a convergent p-series

(a) the radius is $\frac{1}{4}$; the interval of convergence is $1 \le x \le \frac{3}{2}$
(b) the interval of absolute convergence is $1 \le x \le \frac{3}{2}$
(c) there are no values for which the series converges conditionally

32. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(3x+1)^{n+2}}{2n+4}\cdot\frac{2n+2}{(3x+1)^{n+1}}\right| < 1 \Rightarrow |3x+1| \lim_{n\to\infty}\left(\frac{2n+2}{2n+4}\right) < 1 \Rightarrow |3x+1| < 1$
$\Rightarrow -1 < 3x+1 < 1 \Rightarrow -\frac{2}{3} < x < 0$; when $x = -\frac{2}{3}$ we have $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n+1}$, a conditionally convergent series; when $x = 0$ we have $\sum_{n=1}^{\infty}\frac{(1)^{n+1}}{2n+1} = \sum_{n=1}^{\infty}\frac{1}{2n+1}$, a divergent series

(a) the radius is $\frac{1}{3}$; the interval of convergence is $-\frac{2}{3} \le x < 0$
(b) the interval of absolute convergence is $-\frac{2}{3} < x < 0$
(c) the series converges conditionally at $x = -\frac{2}{3}$

33. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{2\cdot4\cdot6\cdots(2n)(2(n+1))} \cdot \frac{2\cdot4\cdot6\cdots(2n)}{x^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \left(\frac{1}{2n+2}\right) < 1$ for all x
(a) the radius is ∞; the series converges for all x
(b) the series converges absolutely for all x
(c) there are no values for which the series converges conditionally

34. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{3\cdot5\cdot7\cdots(2n+1)(2(n+1)+1)x^{n+2}}{(n+1)^2 2^{n+1}} \cdot \frac{n^2 2^n}{3\cdot5\cdot7\cdots(2n+1)x^{n+1}}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \left(\frac{(2n+3)n^2}{2(n+1)^2}\right) < 1 \Rightarrow$ only $x = 0$ satisfies this inequality
(a) the radius is 0; the series converges only for $x = 0$
(b) the series converges absolutely only for $x = 0$
(c) there are no values for which the series converges conditionally

35. For the series $\sum_{n=1}^{\infty} \frac{1+2+\cdots+n}{1^2+2^2+\cdots+n^2} x^n$, recall $1+2+\cdots+n = \frac{n(n+1)}{2}$ and $1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$ so that we can rewrite the series as $\sum_{n=1}^{\infty} \left(\frac{\frac{n(n+1)}{2}}{\frac{n(n+1)(2n+1)}{6}}\right) x^n = \sum_{n=1}^{\infty} \left(\frac{3}{2n+1}\right) x^n$; then $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{3x^{n+1}}{(2(n+1)+1)} \cdot \frac{(2n+1)}{3x^n}\right| < 1$
$\Rightarrow |x| \lim_{n\to\infty} \left|\frac{(2n+1)}{(2n+3)}\right| < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} \left(\frac{3}{2n+1}\right)(-1)^n$, a conditionally convergent series; when $x = 1$ we have $\sum_{n=1}^{\infty} \left(\frac{3}{2n+1}\right)$, a divergent series.
(a) the radius is 1; the interval of convergence is $-1 \le x < 1$
(b) the interval of absolute convergence is $-1 < x < 1$
(c) the series converges conditionally at $x = -1$

36. For the series $\sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)(x-3)^n$, note that $\sqrt{n+1} - \sqrt{n} = \frac{\sqrt{n+1}-\sqrt{n}}{1} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ so that we can rewrite the series as $\sum_{n=1}^{\infty} \frac{(x-3)^n}{\sqrt{n+1}+\sqrt{n}}$; then $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{\sqrt{n+2}+\sqrt{n+1}} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{(x-3)^n}\right| < 1$
$\Rightarrow |x-3| \lim_{n\to\infty} \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n+1}} < 1 \Rightarrow |x-3| < 1 \Rightarrow 2 < x < 4$; when $x = 2$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$, a conditionally convergent series; when $x = 4$ we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$, a divergent series;
(a) the radius is 1; the interval of convergence is $2 \le x < 4$
(b) the interval of absolute convergence is $2 < x < 4$
(c) the series converges conditionally at $x = 2$

37. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{3\cdot6\cdot9\cdots(3n)(3(n+1))} \cdot \frac{3\cdot6\cdot9\cdots(3n)}{n!\,x^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \left|\frac{(n+1)}{3(n+1)}\right| < 1 \Rightarrow \frac{|x|}{3} < 1 \Rightarrow |x| < 3 \Rightarrow R = 3$

38. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(2\cdot4\cdot6\cdots(2n)(2(n+1)))^2 x^{n+1}}{(2\cdot5\cdot8\cdots(3n-1)(3(n+1)-1))^2} \cdot \frac{(2\cdot5\cdot8\cdots(3n-1))^2}{(2\cdot4\cdot6\cdots(2n))^2 x^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \left|\frac{(2n+2)^2}{(3n+2)^2}\right| < 1 \Rightarrow \frac{4|x|}{9} < 1$
$\Rightarrow |x| < \frac{9}{4} \Rightarrow R = \frac{9}{4}$

39. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{((n+1)!)^2 x^{n+1}}{2^{n+1}(2(n+1))!} \cdot \frac{2^n(2n)!}{(n!)^2 x^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \left|\frac{(n+1)^2}{2(2n+2)(2n+1)}\right| < 1 \Rightarrow \frac{|x|}{8} < 1 \Rightarrow |x| < 8 \Rightarrow R = 8$

40. $\lim_{n\to\infty} \sqrt[n]{u_n} < 1 \Rightarrow \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2} x^n} < 1 \Rightarrow |x| \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n < 1 \Rightarrow |x|e^{-1} < 1 \Rightarrow |x| < e \Rightarrow R = e$

41. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{3^{n+1}x^{n+1}}{3^n x^n}\right| < 1 \Rightarrow |x|\lim_{n\to\infty} 3 < 1 \Rightarrow |x| < \frac{1}{3} \Rightarrow -\frac{1}{3} < x < \frac{1}{3}$; at $x = -\frac{1}{3}$ we have $\sum_{n=0}^{\infty} 3^n\left(-\frac{1}{3}\right)^n = \sum_{n=0}^{\infty}(-1)^n$, which diverges; at $x = \frac{1}{3}$ we have $\sum_{n=0}^{\infty} 3^n\left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} 1$, which diverges. The series $\sum_{n=0}^{\infty} 3^n x^n$ $= \sum_{n=0}^{\infty}(3x)^n$ is a convergent geometric series when $-\frac{1}{3} < x < \frac{1}{3}$ and the sum is $\frac{1}{1-3x}$.

42. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(e^x-4)^{n+1}}{(e^x-4)^n}\right| < 1 \Rightarrow |e^x-4|\lim_{n\to\infty} 1 < 1 \Rightarrow |e^x-4| < 1 \Rightarrow 3 < e^x < 5 \Rightarrow \ln 3 < x < \ln 5$; at $x = \ln 3$ we have $\sum_{n=0}^{\infty}\left(e^{\ln 3}-4\right)^n = \sum_{n=0}^{\infty}(-1)^n$, which diverges; at $x = \ln 5$ we have $\sum_{n=0}^{\infty}\left(e^{\ln 5}-4\right)^n = \sum_{n=0}^{\infty} 1$, which diverges. The series $\sum_{n=0}^{\infty}(e^x-4)^n$ is a convergent geometric series when $\ln 3 < x < \ln 5$ and the sum is $\frac{1}{1-(e^x-4)} = \frac{1}{5-e^x}$.

43. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x-1)^{2n+2}}{4^{n+1}}\cdot\frac{4^n}{(x-1)^{2n}}\right| < 1 \Rightarrow \frac{(x-1)^2}{4}\lim_{n\to\infty}|1| < 1 \Rightarrow (x-1)^2 < 4 \Rightarrow |x-1| < 2$ $\Rightarrow -2 < x-1 < 2 \Rightarrow -1 < x < 3$; at $x = -1$ we have $\sum_{n=0}^{\infty}\frac{(-2)^{2n}}{4^n} = \sum_{n=0}^{\infty}\frac{4^n}{4^n} = \sum_{n=0}^{\infty} 1$, which diverges; at $x = 3$ we have $\sum_{n=0}^{\infty}\frac{2^{2n}}{4^n} = \sum_{n=0}^{\infty}\frac{4^n}{4^n} = \sum_{n=0}^{\infty} 1$, a divergent series; the interval of convergence is $-1 < x < 3$; the series $\sum_{n=0}^{\infty}\frac{(x-1)^{2n}}{4^n} = \sum_{n=0}^{\infty}\left(\left(\frac{x-1}{2}\right)^2\right)^n$ is a convergent geometric series when $-1 < x < 3$ and the sum is
$\frac{1}{1-\left(\frac{x-1}{2}\right)^2} = \frac{1}{\left[\frac{4-(x-1)^2}{4}\right]} = \frac{4}{4-x^2+2x-1} = \frac{4}{3+2x-x^2}$

44. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x+1)^{2n+2}}{9^{n+1}}\cdot\frac{9^n}{(x+1)^{2n}}\right| < 1 \Rightarrow \frac{(x+1)^2}{9}\lim_{n\to\infty}|1| < 1 \Rightarrow (x+1)^2 < 9 \Rightarrow |x+1| < 3$ $\Rightarrow -3 < x+1 < 3 \Rightarrow -4 < x < 2$; when $x = -4$ we have $\sum_{n=0}^{\infty}\frac{(-3)^{2n}}{9^n} = \sum_{n=0}^{\infty} 1$ which diverges; at $x = 2$ we have $\sum_{n=0}^{\infty}\frac{3^{2n}}{9^n} = \sum_{n=0}^{\infty} 1$ which also diverges; the interval of convergence is $-4 < x < 2$; the series $\sum_{n=0}^{\infty}\frac{(x+1)^{2n}}{9^n} = \sum_{n=0}^{\infty}\left(\left(\frac{x+1}{3}\right)^2\right)^n$ is a convergent geometric series when $-4 < x < 2$ and the sum is
$\frac{1}{1-\left(\frac{x+1}{3}\right)^2} = \frac{1}{\left[\frac{9-(x+1)^2}{9}\right]} = \frac{9}{9-x^2-2x-1} = \frac{9}{8-2x-x^2}$

45. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{\left(\sqrt{x}-2\right)^{n+1}}{2^{n+1}}\cdot\frac{2^n}{\left(\sqrt{x}-2\right)^n}\right| < 1 \Rightarrow \left|\sqrt{x}-2\right| < 2 \Rightarrow -2 < \sqrt{x}-2 < 2 \Rightarrow 0 < \sqrt{x} < 4$ $\Rightarrow 0 < x < 16$; when $x = 0$ we have $\sum_{n=0}^{\infty}(-1)^n$, a divergent series; when $x = 16$ we have $\sum_{n=0}^{\infty}(1)^n$, a divergent series; the interval of convergence is $0 < x < 16$; the series $\sum_{n=0}^{\infty}\left(\frac{\sqrt{x}-2}{2}\right)^n$ is a convergent geometric series when $0 < x < 16$ and its sum is $\frac{1}{1-\left(\frac{\sqrt{x}-2}{2}\right)} = \frac{1}{\left(\frac{2-\sqrt{x}+2}{2}\right)} = \frac{2}{4-\sqrt{x}}$

46. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(\ln x)^{n+1}}{(\ln x)^n}\right| < 1 \Rightarrow |\ln x| < 1 \Rightarrow -1 < \ln x < 1 \Rightarrow e^{-1} < x < e$; when $x = e^{-1}$ or e we obtain the series $\sum_{n=0}^{\infty} 1^n$ and $\sum_{n=0}^{\infty}(-1)^n$ which both diverge; the interval of convergence is $e^{-1} < x < e$; $\sum_{n=0}^{\infty}(\ln x)^n = \frac{1}{1-\ln x}$ when $e^{-1} < x < e$

47. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\left(\frac{x^2+1}{3}\right)^{n+1} \cdot \left(\frac{3}{x^2+1}\right)^n\right| < 1 \Rightarrow \frac{(x^2+1)}{3} \lim_{n\to\infty} |1| < 1 \Rightarrow \frac{x^2+1}{3} < 1 \Rightarrow x^2 < 2$ $\Rightarrow |x| < \sqrt{2} \Rightarrow -\sqrt{2} < x < \sqrt{2}$; at $x = \pm\sqrt{2}$ we have $\sum_{n=0}^{\infty} (1)^n$ which diverges; the interval of convergence is $-\sqrt{2} < x < \sqrt{2}$; the series $\sum_{n=0}^{\infty} \left(\frac{x^2+1}{3}\right)^n$ is a convergent geometric series when $-\sqrt{2} < x < \sqrt{2}$ and its sum is $\frac{1}{1-\left(\frac{x^2+1}{3}\right)} = \frac{1}{\left(\frac{3-x^2-1}{3}\right)} = \frac{3}{2-x^2}$

48. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x^2-1)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(x^2+1)^n}\right| < 1 \Rightarrow |x^2-1| < 2 \Rightarrow -\sqrt{3} < x < \sqrt{3}$; when $x = \pm\sqrt{3}$ we have $\sum_{n=0}^{\infty} 1^n$, a divergent series; the interval of convergence is $-\sqrt{3} < x < \sqrt{3}$; the series $\sum_{n=0}^{\infty} \left(\frac{x^2-1}{2}\right)^n$ is a convergent geometric series when $-\sqrt{3} < x < \sqrt{3}$ and its sum is $\frac{1}{1-\left(\frac{x^2-1}{2}\right)} = \frac{1}{\left(\frac{2-\left(x^2-1\right)}{2}\right)} = \frac{2}{3-x^2}$

49. $\lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(x-3)^n}\right| < 1 \Rightarrow |x-3| < 2 \Rightarrow 1 < x < 5$; when $x = 1$ we have $\sum_{n=1}^{\infty} (1)^n$ which diverges; when $x = 5$ we have $\sum_{n=1}^{\infty} (-1)^n$ which also diverges; the interval of convergence is $1 < x < 5$; the sum of this convergent geometric series is $\frac{1}{1+\left(\frac{x-3}{2}\right)} = \frac{2}{x-1}$. If $f(x) = 1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \ldots + \left(-\frac{1}{2}\right)^n (x-3)^n + \ldots$ $= \frac{2}{x-1}$ then $f'(x) = -\frac{1}{2} + \frac{1}{2}(x-3) + \ldots + \left(-\frac{1}{2}\right)^n n(x-3)^{n-1} + \ldots$ is convergent when $1 < x < 5$, and diverges when $x = 1$ or 5. The sum for $f'(x)$ is $\frac{-2}{(x-1)^2}$, the derivative of $\frac{2}{x-1}$.

50. If $f(x) = 1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \ldots + \left(-\frac{1}{2}\right)^n (x-3)^n + \ldots = \frac{2}{x-1}$ then $\int f(x)\,dx$ $= x - \frac{(x-3)^2}{4} + \frac{(x-3)^3}{12} + \ldots + \left(-\frac{1}{2}\right)^n \frac{(x-3)^{n+1}}{n+1} + \ldots$. At $x = 1$ the series $\sum_{n=1}^{\infty} \frac{-2}{n+1}$ diverges; at $x = 5$ the series $\sum_{n=1}^{\infty} \frac{(-1)^n 2}{n+1}$ converges. Therefore the interval of convergence is $1 < x \le 5$ and the sum is $2\ln|x-1| + (3 - \ln 4)$, since $\int \frac{2}{x-1}\,dx = 2\ln|x-1| + C$, where $C = 3 - \ln 4$ when $x = 3$.

51. (a) Differentiate the series for sin x to get $\cos x = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \frac{9x^8}{9!} - \frac{11x^{10}}{11!} + \ldots$ $= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots$. The series converges for all values of x since
$$\lim_{n\to\infty} \left|\frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}}\right| = x^2 \lim_{n\to\infty} \left(\frac{1}{(2n+1)(2n+2)}\right) = 0 < 1 \text{ for all x.}$$
(b) $\sin 2x = 2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \frac{2^7x^7}{7!} + \frac{2^9x^9}{9!} - \frac{2^{11}x^{11}}{11!} + \ldots = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \frac{512x^9}{9!} - \frac{2048x^{11}}{11!} + \ldots$
(c)
$$2\sin x\cos x = 2\left[(0\cdot 1) + (0\cdot 0 + 1\cdot 1)x + \left(0\cdot\frac{-1}{2} + 1\cdot 0 + 0\cdot 1\right)x^2 + \left(0\cdot 0 - 1\cdot\frac{1}{2} + 0\cdot 0 - 1\cdot\frac{1}{3!}\right)x^3\right.$$
$$+ \left(0\cdot\frac{1}{4!} + 1\cdot 0 - 0\cdot\frac{1}{2} - 0\cdot\frac{1}{3!} + 0\cdot 1\right)x^4 + \left(0\cdot 0 + 1\cdot\frac{1}{4!} + 0\cdot 0 + \frac{1}{2}\cdot\frac{1}{3!} + 0\cdot 0 + 1\cdot\frac{1}{5!}\right)x^5$$
$$\left.+ \left(0\cdot\frac{1}{6!} + 1\cdot 0 + 0\cdot\frac{1}{4!} + 0\cdot\frac{1}{3!} + 0\cdot\frac{1}{2} + 0\cdot\frac{1}{5!} + 0\cdot 1\right)x^6 + \ldots\right] = 2\left[x - \frac{4x^3}{3!} + \frac{16x^5}{5!} - \ldots\right]$$
$$= 2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \frac{2^7x^7}{7!} + \frac{2^9x^9}{9!} - \frac{2^{11}x^{11}}{11!} + \ldots$$

52. (a) $\frac{d}{x}(e^x) = 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \frac{5x^4}{5!} + \ldots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = e^x$; thus the derivative of e^x is e^x itself
(b) $\int e^x\,dx = e^x + C = x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots + C$, which is the general antiderivative of e^x
(c) $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots$; $e^{-x}\cdot e^x = 1\cdot 1 + (1\cdot 1 - 1\cdot 1)x + \left(1\cdot\frac{1}{2!} - 1\cdot 1 + \frac{1}{2!}\cdot 1\right)x^2$
$$+ \left(1\cdot\frac{1}{3!} - 1\cdot\frac{1}{2!} + \frac{1}{2!}\cdot 1 - \frac{1}{3!}\cdot 1\right)x^3 + \left(1\cdot\frac{1}{4!} - 1\cdot\frac{1}{3!} + \frac{1}{2!}\cdot\frac{1}{2!} - \frac{1}{3!}\cdot 1 + \frac{1}{4!}\cdot 1\right)x^4$$
$$+ \left(1\cdot\frac{1}{5!} - 1\cdot\frac{1}{4!} + \frac{1}{2!}\cdot\frac{1}{3!} - \frac{1}{3!}\cdot\frac{1}{2!} + \frac{1}{4!}\cdot 1 - \frac{1}{5!}\cdot 1\right)x^5 + \ldots = 1 + 0 + 0 + 0 + 0 + 0 + \ldots$$

53. (a) $\ln|\sec x| + C = \int \tan x\,dx = \int \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \ldots\right) dx$
$= \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \frac{31x^{10}}{14{,}175} + \ldots + C;\ x = 0 \Rightarrow C = 0 \Rightarrow \ln|\sec x| = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \frac{31x^{10}}{14{,}175} + \ldots,$
converges when $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(b) $\sec^2 x = \frac{d(\tan x)}{dx} = \frac{d}{dx}\left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \ldots\right) = 1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \frac{62x^8}{315} + \ldots$, converges when $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(c) $\sec^2 x = (\sec x)(\sec x) = \left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots\right)\left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots\right)$
$= 1 + \left(\frac{1}{2} + \frac{1}{2}\right)x^2 + \left(\frac{5}{24} + \frac{1}{4} + \frac{5}{24}\right)x^4 + \left(\frac{61}{720} + \frac{5}{48} + \frac{5}{48} + \frac{61}{720}\right)x^6 + \ldots$
$= 1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \frac{62x^8}{315} + \ldots,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

54. (a) $\ln|\sec x + \tan x| + C = \int \sec x\,dx = \int \left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots\right) dx$
$= x + \frac{x^3}{6} + \frac{x^5}{24} + \frac{61x^7}{5040} + \frac{277x^9}{72{,}576} + \ldots + C;\ x = 0 \Rightarrow C = 0 \Rightarrow \ln|\sec x + \tan x|$
$= x + \frac{x^3}{6} + \frac{x^5}{24} + \frac{61x^7}{5040} + \frac{277x^9}{72{,}576} + \ldots$, converges when $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(b) $\sec x \tan x = \frac{d(\sec x)}{dx} = \frac{d}{dx}\left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots\right) = x + \frac{5x^3}{6} + \frac{61x^5}{120} + \frac{277x^7}{1008} + \ldots$, converges when $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(c) $(\sec x)(\tan x) = \left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots\right)\left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots\right)$
$= x + \left(\frac{1}{3} + \frac{1}{2}\right)x^3 + \left(\frac{2}{15} + \frac{1}{6} + \frac{5}{24}\right)x^5 + \left(\frac{17}{315} + \frac{1}{15} + \frac{5}{72} + \frac{61}{720}\right)x^7 + \ldots = x + \frac{5x^3}{6} + \frac{61x^5}{120} + \frac{277x^7}{1008} + \ldots,$
$-\frac{\pi}{2} < x < \frac{\pi}{2}$

55. (a) If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-(k-1))\,a_n x^{n-k}$ and $f^{(k)}(0) = k!a_k$
$\Rightarrow a_k = \frac{f^{(k)}(0)}{k!}$; likewise if $f(x) = \sum_{n=0}^{\infty} b_n x^n$, then $b_k = \frac{f^{(k)}(0)}{k!} \Rightarrow a_k = b_k$ for every nonnegative integer k

(b) If $f(x) = \sum_{n=0}^{\infty} a_n x^n = 0$ for all x, then $f^{(k)}(x) = 0$ for all x $\Rightarrow$ from part (a) that $a_k = 0$ for every nonnegative integer k

10.8 TAYLOR AND MACLAURIN SERIES

1. $f(x) = e^{2x}, f'(x) = 2e^{2x}, f''(x) = 4e^{2x}, f'''(x) = 8e^{2x}; f(0) = e^{2(0)} = 1, f'(0) = 2, f''(0) = 4, f'''(0) = 8 \Rightarrow P_0(x) = 1,$
$P_1(x) = 1 + 2x, P_2(x) = 1 + x + 2x^2, P_3(x) = 1 + x + 2x^2 + \frac{4}{3}x^3$

2. $f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x; f(0) = \sin 0 = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1$
$\Rightarrow P_0(x) = 0, P_1(x) = x, P_2(x) = x, P_3(x) = x - \frac{1}{6}x^3$

3. $f(x) = \ln x, f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2}, f'''(x) = \frac{2}{x^3}; f(1) = \ln 1 = 0, f'(1) = 1, f''(1) = -1, f'''(1) = 2 \Rightarrow P_0(x) = 0,$
$P_1(x) = (x-1), P_2(x) = (x-1) - \frac{1}{2}(x-1)^2, P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

4. $f(x) = \ln(1+x), f'(x) = \frac{1}{1+x} = (1+x)^{-1}, f''(x) = -(1+x)^{-2}, f'''(x) = 2(1+x)^{-3}; f(0) = \ln 1 = 0,$
$f'(0) = \frac{1}{1} = 1, f''(0) = -(1)^{-2} = -1, f'''(0) = 2(1)^{-3} = 2 \Rightarrow P_0(x) = 0, P_1(x) = x, P_2(x) = x - \frac{x^2}{2}, P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$

5. $f(x) = \frac{1}{x} = x^{-1}, f'(x) = -x^{-2}, f''(x) = 2x^{-3}, f'''(x) = -6x^{-4}; f(2) = \frac{1}{2}, f'(2) = -\frac{1}{4}, f''(2) = \frac{1}{4}, f'''(x) = -\frac{3}{8}$
$\Rightarrow P_0(x) = \frac{1}{2}, P_1(x) = \frac{1}{2} - \frac{1}{4}(x-2), P_2(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2,$
$P_3(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$

6. $f(x) = (x+2)^{-1}$, $f'(x) = -(x+2)^{-2}$, $f''(x) = 2(x+2)^{-3}$, $f'''(x) = -6(x+2)^{-4}$; $f(0) = (2)^{-1} = \frac{1}{2}$, $f'(0) = -(2)^{-2} = -\frac{1}{4}$, $f''(0) = 2(2)^{-3} = \frac{1}{4}$, $f'''(0) = -6(2)^{-4} = -\frac{3}{8} \Rightarrow P_0(x) = \frac{1}{2}$, $P_1(x) = \frac{1}{2} - \frac{x}{4}$, $P_2(x) = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8}$, $P_3(x) = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16}$

7. $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$; $f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$, $f'''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2} \Rightarrow P_0 = \frac{\sqrt{2}}{2}$, $P_1(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$, $P_2(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2$, $P_3(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3$

8. $f(x) = \tan x$, $f'(x) = \sec^2 x$, $f''(x) = 2\sec^2 x \tan x$, $f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$; $f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$, $f'\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = 2$, $f''\left(\frac{\pi}{4}\right) = 2\sec^2\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) = 4$, $f'''\left(\frac{\pi}{4}\right) = 2\sec^4\left(\frac{\pi}{4}\right) + 4\sec^2\left(\frac{\pi}{4}\right)\tan^2\left(\frac{\pi}{4}\right) = 16 \Rightarrow P_0(x) = 1$, $P_1(x) = 1 + 2\left(x - \frac{\pi}{4}\right)$, $P_2(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2$, $P_3(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$

9. $f(x) = \sqrt{x} = x^{1/2}$, $f'(x) = \left(\frac{1}{2}\right)x^{-1/2}$, $f''(x) = \left(-\frac{1}{4}\right)x^{-3/2}$, $f'''(x) = \left(\frac{3}{8}\right)x^{-5/2}$; $f(4) = \sqrt{4} = 2$, $f'(4) = \left(\frac{1}{2}\right)4^{-1/2} = \frac{1}{4}$, $f''(4) = \left(-\frac{1}{4}\right)4^{-3/2} = -\frac{1}{32}$, $f'''(4) = \left(\frac{3}{8}\right)4^{-5/2} = \frac{3}{256} \Rightarrow P_0(x) = 2$, $P_1(x) = 2 + \frac{1}{4}(x-4)$, $P_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$, $P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$

10. $f(x) = (1-x)^{1/2}$, $f'(x) = -\frac{1}{2}(1-x)^{-1/2}$, $f''(x) = -\frac{1}{4}(1-x)^{-3/2}$, $f'''(x) = -\frac{3}{8}(1-x)^{-5/2}$; $f(0) = (1)^{1/2} = 1$, $f'(0) = -\frac{1}{2}(1)^{-1/2} = -\frac{1}{2}$, $f''(0) = -\frac{1}{4}(1)^{-3/2} = -\frac{1}{4}$, $f'''(0) = -\frac{3}{8}(1)^{-5/2} = -\frac{3}{8} \Rightarrow P_0(x) = 1$, $P_1(x) = 1 - \frac{1}{2}x$, $P_2(x) = 1 - \frac{1}{2}x - \frac{1}{8}x^2$, $P_3(x) = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$

11. $f(x) = e^{-x}$, $f'(x) = -e^{-x}$, $f''(x) = e^{-x}$, $f'''(x) = -e^{-x} \Rightarrow \ldots f^{(k)}(x) = (-1)^k e^{-x}$; $f(0) = e^{-(0)} = 1$, $f'(0) = -1$, $f''(0) = 1$, $f'''(0) = -1, \ldots, f^{(k)}(0) = (-1)^k \Rightarrow e^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^n$

12. $f(x) = x\,e^x$, $f'(x) = x\,e^x + e^x$, $f''(x) = x\,e^x + 2e^x$, $f'''(x) = x\,e^x + 3e^x \Rightarrow \ldots f^{(k)}(x) = x\,e^x + k\,e^x$; $f(0) = (0)e^{(0)} = 0$, $f'(0) = 1$, $f''(0) = 2$, $f'''(0) = 3, \ldots, f^{(k)}(0) = k \Rightarrow x + x^2 + \frac{1}{2}x^3 + \ldots = \sum_{n=0}^{\infty} \frac{1}{(n-1)!}x^n$

13. $f(x) = (1+x)^{-1} \Rightarrow f'(x) = -(1+x)^{-2}$, $f''(x) = 2(1+x)^{-3}$, $f'''(x) = -3!(1+x)^{-4} \Rightarrow \ldots f^{(k)}(x) = (-1)^k k!(1+x)^{-k-1}$; $f(0) = 1$, $f'(0) = -1$, $f''(0) = 2$, $f'''(0) = -3!, \ldots, f^{(k)}(0) = (-1)^k k!$ $\Rightarrow 1 - x + x^2 - x^3 + \ldots = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$

14. $f(x) = \frac{2+x}{1-x} \Rightarrow f'(x) = \frac{3}{(1-x)^2}$, $f''(x) = 6(1-x)^{-3}$, $f'''(x) = 18(1-x)^{-4} \Rightarrow \ldots f^{(k)}(x) = 3(k!)(1-x)^{-k-1}$; $f(0) = 2$, $f'(0) = 3$, $f''(0) = 6$, $f'''(0) = 18, \ldots, f^{(k)}(0) = 3(k!) \Rightarrow 2 + 3x + 3x^2 + 3x^3 + \ldots = 2 + \sum_{n=1}^{\infty} 3x^n$

15. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin 3x = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{(2n+1)!} = 3x - \frac{3^3 x^3}{3!} + \frac{3^5 x^5}{5!} - \ldots$

16. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin\frac{x}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}(2n+1)!} = \frac{x}{2} - \frac{x^3}{2^3 \cdot 3!} + \frac{x^5}{2^5 \cdot 5!} + \ldots$

17. $7\cos(-x) = 7\cos x = 7\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 7 - \frac{7x^2}{2!} + \frac{7x^4}{4!} - \frac{7x^6}{6!} + \ldots$, since the cosine is an even function

18. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow 5\cos \pi x = 5\sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!} = 5 - \frac{5\pi^2 x^2}{2!} + \frac{5\pi^4 x^4}{4!} - \frac{5\pi^6 x^6}{6!} + \ldots$

19. $\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left[\left(1 + x^2 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\right)\right] = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$
$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

20. $\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\right)\right] = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots$
$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

21. $f(x) = x^4 - 2x^3 - 5x + 4 \Rightarrow f'(x) = 4x^3 - 6x^2 - 5, f''(x) = 12x^2 - 12x, f'''(x) = 24x - 12, f^{(4)}(x) = 24$
$\Rightarrow f^{(n)}(x) = 0$ if $n \ge 5$; $f(0) = 4, f'(0) = -5, f''(0) = 0, f'''(0) = -12, f^{(4)}(0) = 24, f^{(n)}(0) = 0$ if $n \ge 5$
$\Rightarrow x^4 - 2x^3 - 5x + 4 = 4 - 5x - \frac{12}{3!}x^3 + \frac{24}{4!}x^4 = x^4 - 2x^3 - 5x + 4$

22. $f(x) = \frac{x^2}{x+1} \Rightarrow f'(x) = \frac{2x+x^2}{(x+1)^2}; f''(x) = \frac{2}{(x+1)^3}; f'''(x) = \frac{-6}{(x+1)^4} \Rightarrow f^{(n)}(x) = \frac{(-1)^n n!}{(x+1)^{n+1}}; f(0) = 0, f'(0) = 0, f''(0) = 2,$
$f'''(0) = -6, f^{(n)}(0) = (-1)^n n!$ if $n \ge 2 \Rightarrow x^2 - x^3 + x^4 - x^5 + \ldots = \sum_{n=2}^{\infty} (-1)^n x^n$

23. $f(x) = x^3 - 2x + 4 \Rightarrow f'(x) = 3x^2 - 2, f''(x) = 6x, f'''(x) = 6 \Rightarrow f^{(n)}(x) = 0$ if $n \ge 4$; $f(2) = 8, f'(2) = 10,$
$f''(2) = 12, f'''(2) = 6, f^{(n)}(2) = 0$ if $n \ge 4 \Rightarrow x^3 - 2x + 4 = 8 + 10(x-2) + \frac{12}{2!}(x-2)^2 + \frac{6}{3!}(x-2)^3$
$= 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3$

24. $f(x) = 2x^3 + x^2 + 3x - 8 \Rightarrow f'(x) = 6x^2 + 2x + 3, f''(x) = 12x + 2, f'''(x) = 12 \Rightarrow f^{(n)}(x) = 0$ if $n \ge 4$; $f(1) = -2,$
$f'(1) = 11, f''(1) = 14, f'''(1) = 12, f^{(n)}(1) = 0$ if $n \ge 4 \Rightarrow 2x^3 + x^2 + 3x - 8$
$= -2 + 11(x-1) + \frac{14}{2!}(x-1)^2 + \frac{12}{3!}(x-1)^3 = -2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3$

25. $f(x) = x^4 + x^2 + 1 \Rightarrow f'(x) = 4x^3 + 2x, f''(x) = 12x^2 + 2, f'''(x) = 24x, f^{(4)}(x) = 24, f^{(n)}(x) = 0$ if $n \ge 5$;
$f(-2) = 21, f'(-2) = -36, f''(-2) = 50, f'''(-2) = -48, f^{(4)}(-2) = 24, f^{(n)}(-2) = 0$ if $n \ge 5 \Rightarrow x^4 + x^2 + 1$
$= 21 - 36(x+2) + \frac{50}{2!}(x+2)^2 - \frac{48}{3!}(x+2)^3 + \frac{24}{4!}(x+2)^4 = 21 - 36(x+2) + 25(x+2)^2 - 8(x+2)^3 + (x+2)^4$

26. $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2 \Rightarrow f'(x) = 15x^4 - 4x^3 + 6x^2 + 2x, f''(x) = 60x^3 - 12x^2 + 12x + 2,$
$f'''(x) = 180x^2 - 24x + 12, f^{(4)}(x) = 360x - 24, f^{(5)}(x) = 360, f^{(n)}(x) = 0$ if $n \ge 6$; $f(-1) = -7,$
$f'(-1) = 23, f''(-1) = -82, f'''(-1) = 216, f^{(4)}(-1) = -384, f^{(5)}(-1) = 360, f^{(n)}(-1) = 0$ if $n \ge 6$
$\Rightarrow 3x^5 - x^4 + 2x^3 + x^2 - 2 = -7 + 23(x+1) - \frac{82}{2!}(x+1)^2 + \frac{216}{3!}(x+1)^3 - \frac{384}{4!}(x+1)^4 + \frac{360}{5!}(x+1)^5$
$= -7 + 23(x+1) - 41(x+1)^2 + 36(x+1)^3 - 16(x+1)^4 + 3(x+1)^5$

27. $f(x) = x^{-2} \Rightarrow f'(x) = -2x^{-3}, f''(x) = 3!\,x^{-4}, f'''(x) = -4!\,x^{-5} \Rightarrow f^{(n)}(x) = (-1)^n (n+1)!\,x^{-n-2};$
$f(1) = 1, f'(1) = -2, f''(1) = 3!, f'''(1) = -4!, f^{(n)}(1) = (-1)^n (n+1)! \Rightarrow \frac{1}{x^2}$
$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$

28. $f(x) = \frac{1}{(1-x)^3} \Rightarrow f'(x) = 3(1-x)^{-4}, f''(x) = 12(1-x)^{-5}, f'''(x) = 60(1-x)^{-6} \Rightarrow f^{(n)}(x) = \frac{(n+2)!}{2}(1-x)^{-n-3};$
$f(0) = 1, f'(0) = 3, f''(0) = 12, f'''(0) = 60, \ldots, f^{(n)}(0) = \frac{(n+2)!}{2} \Rightarrow \frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + \ldots$
$= \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^n$

29. $f(x) = e^x \Rightarrow f'(x) = e^x, f''(x) = e^x \Rightarrow f^{(n)}(x) = e^x$; $f(2) = e^2, f'(2) = e^2, \ldots f^{(n)}(2) = e^2$
$\Rightarrow e^x = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^3}{3!}(x-2)^3 + \ldots = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$

30. $f(x) = 2^x \Rightarrow f'(x) = 2^x \ln 2, f''(x) = 2^x(\ln 2)^2, f'''(x) = 2^x(\ln 2)^3 \Rightarrow f^{(n)}(x) = 2^x(\ln 2)^n$; $f(1) = 2, f'(1) = 2\ln 2$, $f''(1) = 2(\ln 2)^2, f'''(1) = 2(\ln 2)^3, \ldots, f^{(n)}(1) = 2(\ln 2)^n$
$\Rightarrow 2^x = 2 + (2\ln 2)(x-1) + \frac{2(\ln 2)^2}{2}(x-1)^2 + \frac{2(\ln 2)^3}{3!}(x-1)^3 + \ldots = \sum_{n=0}^{\infty} \frac{2(\ln 2)^n (x-1)^n}{n!}$

31. $f(x) = \cos\left(2x + \frac{\pi}{2}\right), f'(x) = -2\sin\left(2x + \frac{\pi}{2}\right), f''(x) = -4\cos\left(2x + \frac{\pi}{2}\right), f'''(x) = 8\sin\left(2x + \frac{\pi}{2}\right)$,
$f^{(4)}(x) = 2^4\cos\left(2x + \frac{\pi}{2}\right), f^{(5)}(x) = -2^5\sin\left(2x + \frac{\pi}{2}\right), \ldots; f\left(\frac{\pi}{4}\right) = -1, f'\left(\frac{\pi}{4}\right) = 0, f''\left(\frac{\pi}{4}\right) = 4, f'''\left(\frac{\pi}{4}\right) = 0, f^{(4)}\left(\frac{\pi}{4}\right) = 2^4$,
$f^{(5)}\left(\frac{\pi}{4}\right) = 0, \ldots, f^{(2n)}\left(\frac{\pi}{4}\right) = (-1)^n 2^{2n} \Rightarrow \cos\left(2x + \frac{\pi}{2}\right) = -1 + 2\left(x - \frac{\pi}{4}\right)^2 - \frac{2}{3}\left(x - \frac{\pi}{4}\right)^4 + \ldots$
$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!}\left(x - \frac{\pi}{4}\right)^{2n}$

32. $f(x) = \sqrt{x+1}, f'(x) = \frac{1}{2}(x+1)^{-1/2}, f''(x) = -\frac{1}{4}(x+1)^{-3/2}, f'''(x) = \frac{3}{8}(x+1)^{-5/2}, f^{(4)}(x) = -\frac{15}{16}(x+1)^{-7/2}, \ldots$;
$f(0) = 1, f'(0) = \frac{1}{2}, f''(0) = -\frac{1}{4}, f'''(0) = \frac{3}{8}, f^{(4)}(0) = -\frac{15}{16}, \ldots \Rightarrow \sqrt{x+1} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \ldots$

33. The Maclaurin series generated by $\cos x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$ which converges on $(-\infty, \infty)$ and the Maclaurin series generated by $\frac{2}{1-x}$ is $2\sum_{n=0}^{\infty} x^n$ which converges on $(-1, 1)$. Thus the Maclaurin series generated by $f(x) = \cos x - \frac{2}{1-x}$ is given by $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} - 2\sum_{n=0}^{\infty} x^n = -1 - 2x - \frac{5}{2}x^2 - \ldots$. which converges on the intersection of $(-\infty, \infty)$ and $(-1, 1)$, so the interval of convergence is $(-1, 1)$.

34. The Maclaurin series generated by e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ which converges on $(-\infty, \infty)$. The Maclaurin series generated by $f(x) = (1 - x + x^2)e^x$ is given by $(1 - x + x^2)\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{1}{2}x^2 + \frac{2}{3}x^3 \ldots$. which converges on $(-\infty, \infty)$.

35. The Maclaurin series generated by $\sin x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$ which converges on $(-\infty, \infty)$ and the Maclaurin series generated by $\ln(1+x)$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}x^n$ which converges on $(-1, 1)$. Thus the Maclaurin series genereated by $f(x) = \sin x \cdot \ln(1+x)$ is given by $\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}\right)\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}x^n\right) = x^2 - \frac{1}{2}x^3 + \frac{1}{6}x^4 - \ldots$. which converges on the intersection of $(-\infty, \infty)$ and $(-1, 1)$, so the interval of convergence is $(-1, 1)$.

36. The Maclaurin series generated by $\sin x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$ which converges on $(-\infty, \infty)$. The Maclaurin series genereated by $f(x) = x\sin^2 x$ is given by $x\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}\right)^2 = x\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}\right)$
$= x^3 - \frac{1}{3}x^5 + \frac{2}{45}x^7 + \ldots$ which converges on $(-\infty, \infty)$.

37. If $e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ and $f(x) = e^x$, we have $f^{(n)}(a) = e^a$ f or all $n = 0, 1, 2, 3, \ldots$
$\Rightarrow e^x = e^a\left[\frac{(x-a)^0}{0!} + \frac{(x-a)^1}{1!} + \frac{(x-a)^2}{2!} + \ldots\right] = e^a\left[1 + (x-a) + \frac{(x-a)^2}{2!} + \ldots\right]$ at $x = a$

38. $f(x) = e^x \Rightarrow f^{(n)}(x) = e^x$ for all n $\Rightarrow f^{(n)}(1) = e$ for all $n = 0, 1, 2, \ldots$
$\Rightarrow e^x = e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \ldots = e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \ldots\right]$

39. $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots \Rightarrow f'(x)$
$= f'(a) + f''(a)(x-a) + \frac{f'''(a)}{3!}3(x-a)^2 + \ldots \Rightarrow f''(x) = f''(a) + f'''(a)(x-a) + \frac{f^{(4)}(a)}{4!}4 \cdot 3(x-a)^2 + \ldots$
$\Rightarrow f^{(n)}(x) = f^{(n)}(a) + f^{(n+1)}(a)(x-a) + \frac{f^{(n+2)}(a)}{2}(x-a)^2 + \ldots$
$\Rightarrow f(a) = f(a) + 0, f'(a) = f'(a) + 0, \ldots, f^{(n)}(a) = f^{(n)}(a) + 0$

40. $E(x) = f(x) - b_0 - b_1(x-a) - b_2(x-a)^2 - b_3(x-a)^3 - \ldots - b_n(x-a)^n$
$\Rightarrow 0 = E(a) = f(a) - b_0 \Rightarrow b_0 = f(a)$; from condition (b),
$\lim_{x\to a} \frac{f(x) - f(a) - b_1(x-a) - b_2(x-a)^2 - b_3(x-a)^3 - \ldots - b_n(x-a)^n}{(x-a)^n} = 0$
$\Rightarrow \lim_{x\to a} \frac{f'(x) - b_1 - 2b_2(x-a) - 3b_3(x-a)^2 - \ldots - nb_n(x-a)^{n-1}}{n(x-a)^{n-1}} = 0$
$\Rightarrow b_1 = f'(a) \Rightarrow \lim_{x\to a} \frac{f''(x) - 2b_2 - 3!\,b_3(x-a) - \ldots - n(n-1)b_n(x-a)^{n-2}}{n(n-1)(x-a)^{n-2}} = 0$
$\Rightarrow b_2 = \frac{1}{2}f''(a) \Rightarrow \lim_{x\to a} \frac{f'''(x) - 3!\,b_3 - \ldots - n(n-1)(n-2)b_n(x-a)^{n-3}}{n(n-1)(n-2)(x-a)^{n-3}} = 0$
$= b_3 = \frac{1}{3!}f'''(a) \Rightarrow \lim_{x\to a} \frac{f^{(n)}(x) - n!\,b_n}{n!} = 0 \Rightarrow b_n = \frac{1}{n!}f^{(n)}(a)$; therefore,
$g(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n = P_n(x)$

41. $f(x) = \ln(\cos x) \Rightarrow f'(x) = -\tan x$ and $f''(x) = -\sec^2 x$; $f(0) = 0, f'(0) = 0, f''(0) = -1 \Rightarrow L(x) = 0$ and $Q(x) = -\frac{x^2}{2}$

42. $f(x) = e^{\sin x} \Rightarrow f'(x) = (\cos x)e^{\sin x}$ and $f''(x) = (-\sin x)e^{\sin x} + (\cos x)^2 e^{\sin x}$; $f(0) = 1, f'(0) = 1, f''(0) = 1$
$\Rightarrow L(x) = 1 + x$ and $Q(x) = 1 + x + \frac{x^2}{2}$

43. $f(x) = \left(1 - x^2\right)^{-1/2} \Rightarrow f'(x) = x\left(1 - x^2\right)^{-3/2}$ and $f''(x) = \left(1 - x^2\right)^{-3/2} + 3x^2\left(1 - x^2\right)^{-5/2}$; $f(0) = 1, f'(0) = 0,$
$f''(0) = 1 \Rightarrow L(x) = 1$ and $Q(x) = 1 + \frac{x^2}{2}$

44. $f(x) = \cosh x \Rightarrow f'(x) = \sinh x$ and $f''(x) = \cosh x$; $f(0) = 1, f'(0) = 0, f''(0) = 1 \Rightarrow L(x) = 1$ and $Q(x) = 1 + \frac{x^2}{2}$

45. $f(x) = \sin x \Rightarrow f'(x) = \cos x$ and $f''(x) = -\sin x$; $f(0) = 0, f'(0) = 1, f''(0) = 0 \Rightarrow L(x) = x$ and $Q(x) = x$

46. $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$ and $f''(x) = 2\sec^2 x \tan x$; $f(0) = 0, f'(0) = 1, f'' = 0 \Rightarrow L(x) = x$ and $Q(x) = x$

10.9 CONVERGENCE OF TAYLOR SERIES

1. $e^x = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-5x} = 1 + (-5x) + \frac{(-5x)^2}{2!} + \ldots = 1 - 5x + \frac{5^2x^2}{2!} - \frac{5^3x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n 5^n x^n}{n!}$

2. $e^x = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x/2} = 1 + \left(\frac{-x}{2}\right) + \frac{\left(-\frac{x}{2}\right)^2}{2!} + \ldots = 1 - \frac{x}{2} + \frac{x^2}{2^2 2!} - \frac{x^3}{2^3 3!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n n!}$

3. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow 5\sin(-x) = 5\left[(-x) - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \ldots\right] = \sum_{n=0}^{\infty} \frac{5(-1)^{n+1} x^{2n+1}}{(2n+1)!}$

4. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin\frac{\pi x}{2} = \frac{\pi x}{2} - \frac{\left(\frac{\pi x}{2}\right)^3}{3!} + \frac{\left(\frac{\pi x}{2}\right)^5}{5!} - \frac{\left(\frac{\pi x}{2}\right)^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1} x^{2n+1}}{2^{2n+1}(2n+1)!}$

5. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos 5x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n \left[5x^2\right]^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n} x^{4n}}{(2n)!} = 1 - \frac{25x^4}{2!} + \frac{625x^8}{4!} - \frac{15625x^{12}}{6!} + \ldots$

6. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos\left(\frac{x^{3/2}}{\sqrt{2}}\right) = \cos\left(\left(\frac{x^3}{2}\right)^{1/2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\left(\frac{x^3}{2}\right)^{1/2}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{2^n (2n)!}$
$= 1 - \frac{x^3}{2 \cdot 2!} + \frac{x^6}{2^2 \cdot 4!} - \frac{x^9}{2^3 \cdot 6!} + \ldots$

7. $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \Rightarrow \ln(1+x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x^2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n} = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots$

8. $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \Rightarrow \tan^{-1}(3x^4) = \sum_{n=0}^{\infty} \frac{(-1)^n (3x^4)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{8n+4}}{n} = 3x^4 - 9x^{12} + \frac{243}{5}x^{20} - \frac{2187}{7}x^{28} + \ldots$

9. $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \Rightarrow \frac{1}{1+\frac{3}{4}x^3} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{4}x^3\right)^n = \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{4}\right)^n x^{3n} = 1 - \frac{3}{4}x^3 + \frac{9}{16}x^6 - \frac{27}{64}x^9 + \ldots$

10. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow \frac{1}{2-x} = \frac{1}{2} \frac{1}{1-\frac{1}{2}x} = \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{1}{2}x\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} x^n = \frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots$

11. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow xe^x = x\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \ldots$

12. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow x^2 \sin x = x^2\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!} = x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \ldots$

13. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \frac{x^2}{2} - 1 + \cos x = \frac{x^2}{2} - 1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \frac{x^2}{2} - 1 + 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots$
$= \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots = \sum_{n=2}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

14. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin x - x + \frac{x^3}{3!} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right) - x + \frac{x^3}{3!}$
$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots\right) - x + \frac{x^3}{3!} = \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots = \sum_{n=2}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

15. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow x \cos \pi x = x \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} x^{2n+1}}{(2n)!} = x - \frac{\pi^2 x^3}{2!} + \frac{\pi^4 x^5}{4!} - \frac{\pi^6 x^7}{6!} + \ldots$

16. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow x^2 \cos(x^2) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n)!} = x^2 - \frac{x^6}{2!} + \frac{x^{10}}{4!} - \frac{x^{14}}{6!} + \ldots$

17. $\cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2} = \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \frac{1}{2} + \frac{1}{2}\left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \ldots\right]$
$= 1 - \frac{(2x)^2}{2 \cdot 2!} + \frac{(2x)^4}{2 \cdot 4!} - \frac{(2x)^6}{2 \cdot 6!} + \frac{(2x)^8}{2 \cdot 8!} - \ldots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{2 \cdot (2n)!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!}$

18. $\sin^2 x = \left(\frac{1-\cos 2x}{2}\right) = \frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \ldots\right) = \frac{(2x)^2}{2 \cdot 2!} - \frac{(2x)^4}{2 \cdot 4!} + \frac{(2x)^6}{2 \cdot 6!} - \ldots$
$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2x)^{2n}}{2 \cdot (2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!}$

19. $\frac{x^2}{1-2x} = x^2\left(\frac{1}{1-2x}\right) = x^2 \sum\limits_{n=0}^{\infty} (2x)^n = \sum\limits_{n=0}^{\infty} 2^n x^{n+2} = x^2 + 2x^3 + 2^2x^4 + 2^3x^5 + \ldots$

20. $x\ln(1+2x) = x\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}(2x)^n}{n} = \sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}2^n x^{n+1}}{n} = 2x^2 - \frac{2^2x^3}{2} + \frac{2^3x^4}{4} - \frac{2^4x^5}{5} + \ldots$

21. $\frac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \Rightarrow \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots = \sum\limits_{n=1}^{\infty} nx^{n-1} = \sum\limits_{n=0}^{\infty} (n+1)x^n$

22. $\frac{2}{(1-x)^3} = \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{d}{dx}(1 + 2x + 3x^2 + \ldots) = 2 + 6x + 12x^2 + \ldots = \sum\limits_{n=2}^{\infty} n(n-1)x^{n-2}$
$= \sum\limits_{n=0}^{\infty} (n+2)(n+1)x^n$

23. $\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots \Rightarrow x\tan^{-1}x^2 = x\left(x^2 - \frac{1}{3}(x^2)^3 + \frac{1}{5}(x^2)^5 - \frac{1}{7}(x^2)^7 + \ldots\right)$
$= x^3 - \frac{1}{3}x^7 + \frac{1}{5}x^{11} - \frac{1}{7}x^{15} + \ldots = \sum\limits_{n=1}^{\infty} \frac{(-1)^n x^{4n-1}}{2n-1}$

24. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \Rightarrow \sin x \cdot \cos x = \frac{1}{2}\sin 2x = \frac{1}{2}\left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \ldots\right)$
$= x - \frac{4x^3}{3!} + \frac{16x^5}{5!} - \frac{64x^7}{7!} + \ldots = x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} + \ldots = \sum\limits_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n+1}}{(2n+1)!}$

25. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ and $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots \Rightarrow e^x + \frac{1}{1+x}$
$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) + (1 - x + x^2 - x^3 + \ldots) = 2 + \frac{3}{2}x^2 - \frac{5}{6}x^3 + \frac{25}{24}x^4 + \ldots = \sum\limits_{n=0}^{\infty}\left(\frac{1}{n!} + (-1)^n\right)x^n$

26. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ and $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \Rightarrow \cos x - \sin x$
$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) = 1 - x - \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} - \frac{x^6}{6!} + \frac{x^7}{7!} + \ldots$
$= \sum\limits_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n}}{(2n)!} - \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)$

27. $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots \Rightarrow \frac{x}{3}\ln(1+x^2) = \frac{x}{3}\left(x^2 - \frac{1}{2}(x^2)^2 + \frac{1}{3}(x^2)^3 - \frac{1}{4}(x^2)^4 + \ldots\right)$
$= \frac{1}{3}x^3 - \frac{1}{6}x^5 + \frac{1}{9}x^7 - \frac{1}{12}x^9 + \ldots = \sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n} x^{2n+1}$

28. $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$ and $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots \Rightarrow \ln(1+x) - \ln(1-x)$
$= \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots\right) - \left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots\right) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \ldots = \sum\limits_{n=0}^{\infty} \frac{2}{2n+1} x^{2n+1}$

29. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ and $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \Rightarrow e^x \cdot \sin x$
$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) = x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 - \ldots.$

30. $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$ and $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \Rightarrow \frac{\ln(1+x)}{1-x} = \ln(1+x) \cdot \frac{1}{1-x}$
$= \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots\right)(1 + x + x^2 + x^3 + \ldots) = x + \frac{1}{2}x^2 + \frac{5}{6}x^3 + \frac{7}{12}x^4 + \ldots.$

31. $\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots \Rightarrow (\tan^{-1}x)^2 = (\tan^{-1}x)(\tan^{-1}x)$
$= \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots\right)\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots\right) = x^2 - \frac{2}{3}x^4 - \frac{23}{45}x^6 - \frac{44}{105}x^8 + \ldots.$

32. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ and $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \Rightarrow \cos^2 x \cdot \sin x = \cos x \cdot \cos x \cdot \sin x$
$= \cos x \cdot \frac{1}{2}\sin 2x = \frac{1}{2}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right)\left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \ldots\right) = x - \frac{7}{6}x^3 + \frac{61}{120}x^5 - \frac{1247}{5040}x^7 + \ldots$

33. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ and $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$
$\Rightarrow e^{\sin x} = 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) + \frac{1}{2}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)^3 + \ldots$
$= 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \ldots.$

34. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ and $\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots \Rightarrow \sin(\tan^{-1}x) = \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots\right)$
$- \frac{1}{6}\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots\right)^3 + \frac{1}{120}\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots\right)^5 - \frac{1}{5040}\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots\right)^7 + \ldots$
$= x - \frac{1}{2}x^3 + \frac{3}{8}x^5 - \frac{5}{16}x^7 + \ldots$

35. Since $n = 3$, then $f^{(4)}(x) = \sin x$, $|f^{(4)}(x)| \le M$ on $[0, 0.1] \Rightarrow |\sin x| \le 1$ on $[0, 0.1] \Rightarrow M = 1$. Then $|R_3(0.1)| \le 1\frac{|0.1 - 0|^4}{4!}$
$= 4.2 \times 10^{-6} \Rightarrow$ error $\le 4.2 \times 10^{-6}$

36. Since $n = 4$, then $f^{(5)}(x) = e^x$, $|f^{(5)}(x)| \le M$ on $[0, 0.5] \Rightarrow |e^x| \le \sqrt{e}$ on $[0, 0.5] \Rightarrow M = 2.7$. Then
$|R_4(0.5)| \le 2.7\frac{|0.5 - 0|^5}{5!} = 7.03 \times 10^{-4} \Rightarrow$ error $\le 7.03 \times 10^{-4}$

37. By the Alternating Series Estimation Theorem, the error is less than $\frac{|x|^5}{5!} \Rightarrow |x|^5 < (5!)\,(5 \times 10^{-4}) \Rightarrow |x|^5 < 600 \times 10^{-4}$
$\Rightarrow |x| < \sqrt[5]{6 \times 10^{-2}} \approx 0.56968$

38. If $\cos x = 1 - \frac{x^2}{2}$ and $|x| < 0.5$, then the error is less than $\left|\frac{(.5)^4}{24}\right| = 0.0026$, by Alternating Series Estimation Theorem; since the next term in the series is positive, the approximation $1 - \frac{x^2}{2}$ is too small, by the Alternating Series Estimation Theorem

39. If $\sin x = x$ and $|x| < 10^{-3}$, then the error is less than $\frac{(10^{-3})^3}{3!} \approx 1.67 \times 10^{-10}$, by Alternating Series Estimation Theorem; The Alternating Series Estimation Theorem says $R_2(x)$ has the same sign as $-\frac{x^3}{3!}$. Moreover, $x < \sin x$
$\Rightarrow 0 < \sin x - x = R_2(x) \Rightarrow x < 0 \Rightarrow -10^{-3} < x < 0.$

40. $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \ldots$. By the Alternating Series Estimation Theorem the $|\text{error}| < \left|\frac{-x^2}{8}\right| < \frac{(0.01)^2}{8}$
$= 1.25 \times 10^{-5}$

41. $|R_2(x)| = \left|\frac{e^c x^3}{3!}\right| < \frac{3^{(0.1)}(0.1)^3}{3!} < 1.87 \times 10^{-4}$, where c is between 0 and x

42. $|R_2(x)| = \left|\frac{e^c x^3}{3!}\right| < \frac{(0.1)^3}{3!} = 1.67 \times 10^{-4}$, where c is between 0 and x

43. $\sin^2 x = \left(\frac{1 - \cos 2x}{2}\right) = \frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \ldots\right) = \frac{2x^2}{2!} - \frac{2^3x^4}{4!} + \frac{2^5x^6}{6!} - \ldots$
$\Rightarrow \frac{d}{dx}(\sin^2 x) = \frac{d}{dx}\left(\frac{2x^2}{2!} - \frac{2^3x^4}{4!} + \frac{2^5x^6}{6!} - \ldots\right) = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \ldots \Rightarrow 2\sin x \cos x$
$= 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \ldots = \sin 2x$, which checks

44. $\cos^2 x = \cos 2x + \sin^2 x = \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} + \ldots\right) + \left(\frac{2x^2}{2!} - \frac{2^3x^4}{4!} + \frac{2^5x^6}{6!} - \frac{2^7x^8}{8!} + \ldots\right)$
$= 1 - \frac{2x^2}{2!} + \frac{2^3x^4}{4!} - \frac{2^5x^6}{6!} + \ldots = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \frac{1}{315}x^8 - \ldots$

45. A special case of Taylor's Theorem is $f(b) = f(a) + f'(c)(b-a)$, where c is between a and b $\Rightarrow f(b) - f(a) = f'(c)(b-a)$, the Mean Value Theorem.

46. If f(x) is twice differentiable and at $x = a$ there is a point of inflection, then $f''(a) = 0$. Therefore, $L(x) = Q(x) = f(a) + f'(a)(x-a)$.

47. (a) $f'' \le 0$, $f'(a) = 0$ and $x = a$ interior to the interval I $\Rightarrow f(x) - f(a) = \frac{f''(c_2)}{2}(x-a)^2 \le 0$ throughout I
$\Rightarrow f(x) \le f(a)$ throughout I $\Rightarrow$ f has a local maximum at $x = a$
(b) similar reasoning gives $f(x) - f(a) = \frac{f''(c_2)}{2}(x-a)^2 \ge 0$ throughout I $\Rightarrow f(x) \ge f(a)$ throughout I $\Rightarrow$ f has a local minimum at $x = a$

48. $f(x) = (1-x)^{-1} \Rightarrow f'(x) = (1-x)^{-2} \Rightarrow f''(x) = 2(1-x)^{-3} \Rightarrow f^{(3)}(x) = 6(1-x)^{-4}$
$\Rightarrow f^{(4)}(x) = 24(1-x)^{-5}$; therefore $\frac{1}{1-x} \approx 1 + x + x^2 + x^3$. $|x| < 0.1 \Rightarrow \frac{10}{11} < \frac{1}{1-x} < \frac{10}{9} \Rightarrow \left|\frac{1}{(1-x)^5}\right| < \left(\frac{10}{9}\right)^5$
$\Rightarrow \left|\frac{x^4}{(1-x)^5}\right| < x^4\left(\frac{10}{9}\right)^5 \Rightarrow$ the error $e_3 \le \left|\frac{\max f^{(4)}(x)\, x^4}{4!}\right| < (0.1)^4\left(\frac{10}{9}\right)^5 = 0.00016935 < 0.00017$, since $\left|\frac{f^{(4)}(x)}{4!}\right| = \left|\frac{1}{(1-x)^5}\right|$.

49. (a) $f(x) = (1+x)^k \Rightarrow f'(x) = k(1+x)^{k-1} \Rightarrow f''(x) = k(k-1)(1+x)^{k-2}$; $f(0) = 1$, $f'(0) = k$, and $f''(0) = k(k-1)$
$\Rightarrow Q(x) = 1 + kx + \frac{k(k-1)}{2}x^2$
(b) $|R_2(x)| = \left|\frac{3\cdot2\cdot1}{3!}x^3\right| < \frac{1}{100} \Rightarrow |x^3| < \frac{1}{100} \Rightarrow 0 < x < \frac{1}{100^{1/3}}$ or $0 < x < .21544$

50. (a) Let $P = x + \pi \Rightarrow |x| = |P - \pi| < .5 \times 10^{-n}$ since P approximates π accurate to n decimals. Then,
$P + \sin P = (\pi + x) + \sin(\pi + x) = (\pi + x) - \sin x = \pi + (x - \sin x) \Rightarrow |(P + \sin P) - \pi|$
$= |\sin x - x| \le \frac{|x|^3}{3!} < \frac{0.125}{3!} \times 10^{-3n} < .5 \times 10^{-3n} \Rightarrow P + \sin P$ gives an approximation to π correct to 3n decimals.

51. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)a_n x^{n-k}$ and $f^{(k)}(0) = k!\,a_k$
$\Rightarrow a_k = \frac{f^{(k)}(0)}{k!}$ for k a nonnegative integer. Therefore, the coefficients of f(x) are identical with the corresponding coefficients in the Maclaurin series of f(x) and the statement follows.

52. Note: f even $\Rightarrow f(-x) = f(x) \Rightarrow -f'(-x) = f'(x) \Rightarrow f'(-x) = -f'(x) \Rightarrow f'$ odd;
f odd $\Rightarrow f(-x) = -f(x) \Rightarrow -f'(-x) = -f'(x) \Rightarrow f'(-x) = f'(x) \Rightarrow f'$ even;
also, f odd $\Rightarrow f(-0) = f(0) \Rightarrow 2f(0) = 0 \Rightarrow f(0) = 0$
(a) If f(x) is even, then any odd-order derivative is odd and equal to 0 at $x = 0$. Therefore, $a_1 = a_3 = a_5 = \ldots = 0$; that is, the Maclaurin series for f contains only even powers.
(b) If f(x) is odd, then any even-order derivative is odd and equal to 0 at $x = 0$. Therefore, $a_0 = a_2 = a_4 = \ldots = 0$; that is, the Maclaurin series for f contains only odd powers.

53-58. Example CAS commands:
Maple:
```
f := x -> 1/sqrt(1+x);
x0 := -3/4;
x1 := 3/4;
# Step 1:
plot( f(x), x=x0..x1, title="Step 1: #53 (Section 10.9)" );
```

```
# Step 2:
P1 := unapply( TaylorApproximation(f(x), x = 0, order=1), x );
P2 := unapply( TaylorApproximation(f(x), x = 0, order=2), x );
P3 := unapply( TaylorApproximation(f(x), x = 0, order=3), x );
# Step 3:
D2f := D(D(f));
D3f := D(D(D(f)));
D4f := D(D(D(D(f))));
plot( [D2f(x),D3f(x),D4f(x)], x=x0..x1, thickness=[0,2,4], color=[red,blue,green], title="Step 3: #57 (Section 9.9)" );
c1 := x0;
M1 := abs( D2f(c1) );
c2 := x0;
M2 := abs( D3f(c2) );
c3 := x0;
M3 := abs( D4f(c3) );
# Step 4:
R1 := unapply( abs(M1/2!*(x-0)^2), x );
R2 := unapply( abs(M2/3!*(x-0)^3), x );
R3 := unapply( abs(M3/4!*(x-0)^4), x );
plot( [R1(x),R2(x),R3(x)], x=x0..x1, thickness=[0,2,4], color=[red,blue,green], title="Step 4: #53 (Section 10.9)" );
# Step 5:
E1 := unapply( abs(f(x)-P1(x)), x );
E2 := unapply( abs(f(x)-P2(x)), x );
E3 := unapply( abs(f(x)-P3(x)), x );
plot( [E1(x),E2(x),E3(x),R1(x),R2(x),R3(x)], x=x0..x1, thickness=[0,2,4], color=[red,blue,green],
      linestyle=[1,1,1,3,3,3], title="Step 5: #53 (Section 10.9)" );
# Step 6:
TaylorApproximation( f(x), view=[x0..x1,DEFAULT], x=0, output=animation, order=1..3 );
L1 := fsolve( abs(f(x)-P1(x))=0.01, x=x0/2 );          # (a)
R1 := fsolve( abs(f(x)-P1(x))=0.01, x=x1/2 );
L2 := fsolve( abs(f(x)-P2(x))=0.01, x=x0/2 );
R2 := fsolve( abs(f(x)-P2(x))=0.01, x=x1/2 );
L3 := fsolve( abs(f(x)-P3(x))=0.01, x=x0/2 );
R3 := fsolve( abs(f(x)-P3(x))=0.01, x=x1/2 );
plot( [E1(x),E2(x),E3(x),0.01], x=min(L1,L2,L3)..max(R1,R2,R3), thickness=[0,2,4,0], linestyle=[0,0,0,2],
     color=[red,blue,green,black], view=[DEFAULT,0..0.01], title="#53(a) (Section 10.9)" );
abs(`f(x)`-`P`[1](x) ) <= evalf( E1(x0) );              # (b)
abs(`f(x)`-`P`[2](x) ) <= evalf( E2(x0) );
abs(`f(x)`-`P`[3](x) ) <= evalf( E3(x0) );
```

Mathematica: (assigned function and values for a, b, c, and n may vary)

```
Clear[x, f, c]
f[x_]= (1 + x)^(3/2)
{a, b}= {−1/2, 2};
pf=Plot[ f[x], {x, a, b}];
poly1[x_]=Series[f[x], {x,0,1}]//Normal
poly2[x_]=Series[f[x], {x,0,2}]//Normal
poly3[x_]=Series[f[x], {x,0,3}]//Normal
Plot[{f[x], poly1[x], poly2[x], poly3[x]}, {x, a, b},
       PlotStyle → {RGBColor[1,0,0], RGBColor[0,1,0], RGBColor[0,0,1], RGBColor[0,.5,.5]}];
```

The above defines the approximations. The following analyzes the derivatives to determine their maximum values.

```
f''[c]
Plot[f''[x], {x, a, b}];
f'''[c]
Plot[f'''[x], {x, a, b}];
f''''[c]
Plot[f''''[x], {x, a, b}];
```

Noting the upper bound for each of the above derivatives occurs at x = a, the upper bounds m1, m2, and m3 can be defined and bounds for remainders viewed as functions of x.

```
m1=f''[a]
m2=-f'''[a]
m3=f''''[a]
r1[x_]=m1 x^2 /2!
Plot[r1[x], {x, a, b}];
r2[x_]=m2 x^3 /3!
Plot[r2[x], {x, a, b}];
r3[x_]=m3 x^4 /4!
Plot[r3[x], {x, a, b}];
```

A three dimensional look at the error functions, allowing both c and x to vary can also be viewed. Recall that c must be a value between 0 and x, so some points on the surfaces where c is not in that interval are meaningless.

```
Plot3D[f''[c] x^2 /2!, {x, a, b}, {c, a, b}, PlotRange → All]
Plot3D[f'''[c] x^3 /3!, {x, a, b}, {c, a, b}, PlotRange → All]
Plot3D[f''''[c] x^4 /4!, {x, a, b}, {c, a, b}, PlotRange → All]
```

10.10 THE BINOMIAL SERIES

1. $(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^3}{3!} + \ldots = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \ldots$

2. $(1+x)^{1/3} = 1 + \frac{1}{3}x + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)x^2}{2!} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)x^3}{3!} + \ldots = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \ldots$

3. $(1-x)^{-1/2} = 1 - \frac{1}{2}(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-x)^3}{3!} + \ldots = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots$

4. $(1-2x)^{1/2} = 1 + \frac{1}{2}(-2x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(-2x)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-2x)^3}{3!} + \ldots = 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \ldots$

5. $\left(1+\frac{x}{2}\right)^{-2} = 1 - 2\left(\frac{x}{2}\right) + \frac{(-2)(-3)\left(\frac{x}{2}\right)^2}{2!} + \frac{(-2)(-3)(-4)\left(\frac{x}{2}\right)^3}{3!} + \ldots = 1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3$

6. $\left(1-\frac{x}{3}\right)^4 = 1 + 4\left(-\frac{x}{3}\right) + \frac{(4)(3)\left(-\frac{x}{3}\right)^2}{2!} + \frac{(4)(3)(2)\left(-\frac{x}{3}\right)^3}{3!} + \frac{(4)(3)(2)(1)\left(-\frac{x}{3}\right)^4}{4!} + 0 + \ldots = 1 - \frac{4}{3}x + \frac{2}{3}x^2 - \frac{4}{27}x^3 + \frac{1}{81}x^4$

7. $(1+x^3)^{-1/2} = 1 - \frac{1}{2}x^3 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x^3)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(x^3)^3}{3!} + \ldots = 1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9 + \ldots$

8. $(1+x^2)^{-1/3} = 1 - \frac{1}{3}x^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)(x^2)^2}{2!} + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)(x^2)^3}{3!} + \ldots = 1 - \frac{1}{3}x^2 + \frac{2}{9}x^4 - \frac{14}{81}x^6 + \ldots$

9. $\left(1+\frac{1}{x}\right)^{1/2} = 1 + \frac{1}{2}\left(\frac{1}{x}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{x}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{1}{x}\right)^3}{3!} + \ldots = 1 + \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{16x^3} + \ldots$

10. $\frac{x}{\sqrt[3]{1+x}} = x(1+x)^{-1/3} = x\left(1 - \left(-\frac{1}{3}\right)x + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)x^2}{2!} + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)x^3}{3!} + \ldots\right) = x - \frac{1}{3}x^2 + \frac{2}{9}x^3 - \frac{14}{81}x^4 + \ldots$

11. $(1+x)^4 = 1 + 4x + \frac{(4)(3)x^2}{2!} + \frac{(4)(3)(2)x^3}{3!} + \frac{(4)(3)(2)x^4}{4!} = 1 + 4x + 6x^2 + 4x^3 + x^4$

12. $(1+x^2)^3 = 1 + 3x^2 + \frac{(3)(2)\left(x^2\right)^2}{2!} + \frac{(3)(2)(1)\left(x^2\right)^3}{3!} = 1 + 3x^2 + 3x^4 + x^6$

13. $(1-2x)^3 = 1 + 3(-2x) + \frac{(3)(2)(-2x)^2}{2!} + \frac{(3)(2)(1)(-2x)^3}{3!} = 1 - 6x + 12x^2 - 8x^3$

14. $\left(1 - \frac{x}{2}\right)^4 = 1 + 4\left(-\frac{x}{2}\right) + \frac{(4)(3)\left(-\frac{x}{2}\right)^2}{2!} + \frac{(4)(3)(2)\left(-\frac{x}{2}\right)^3}{3!} + \frac{(4)(3)(2)(1)\left(-\frac{x}{2}\right)^4}{4!} = 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4$

15. $\int_0^{0.2} \sin x^2\, dx = \int_0^{0.2}\left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \ldots\right) dx = \left[\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \ldots\right]_0^{0.2} \approx \left[\frac{x^3}{3}\right]_0^{0.2} \approx 0.00267$ with error $|E| \le \frac{(.2)^7}{7\cdot 3!} \approx 0.0000003$

16. $\int_0^{0.2} \frac{e^{-x}-1}{x}\, dx = \int_0^{0.2} \frac{1}{x}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots - 1\right) dx = \int_0^{0.2}\left(-1 + \frac{x}{2} - \frac{x^2}{6} + \frac{x^3}{24} - \ldots\right) dx$
$= \left[-x + \frac{x^2}{4} - \frac{x^3}{18} + \ldots\right]_0^{0.2} \approx -0.19044$ with error $|E| \le \frac{(0.2)^4}{96} \approx 0.00002$

17. $\int_0^{0.1} \frac{1}{\sqrt{1+x^4}}\, dx = \int_0^{0.1}\left(1 - \frac{x^4}{2} + \frac{3x^8}{8} - \ldots\right) dx = \left[x - \frac{x^5}{10} + \ldots\right]_0^{0.1} \approx [x]_0^{0.1} \approx 0.1$ with error $|E| \le \frac{(0.1)^5}{10} = 0.000001$

18. $\int_0^{0.25} \sqrt[3]{1+x^2}\, dx = \int_0^{0.25}\left(1 + \frac{x^2}{3} - \frac{x^4}{9} + \ldots\right) dx = \left[x + \frac{x^3}{9} - \frac{x^5}{45} + \ldots\right]_0^{0.25} \approx \left[x + \frac{x^3}{9}\right]_0^{0.25} \approx 0.25174$ with error $|E| \le \frac{(0.25)^5}{45} \approx 0.0000217$

19. $\int_0^{0.1} \frac{\sin x}{x}\, dx = \int_0^{0.1}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots\right) dx = \left[x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \ldots\right]_0^{0.1} \approx \left[x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!}\right]_0^{0.1}$
≈ 0.0999444611, $|E| \le \frac{(0.1)^7}{7\cdot 7!} \approx 2.8 \times 10^{-12}$

20. $\int_0^{0.1} \exp\left(-x^2\right) dx = \int_0^{0.1}\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \ldots\right) dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} + \ldots\right]_0^{0.1} \approx \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}\right]_0^{0.1}$
≈ 0.0996676643, $|E| \le \frac{(0.1)^9}{216} \approx 4.6 \times 10^{-12}$

21. $(1+x^4)^{1/2} = (1)^{1/2} + \frac{\left(\frac{1}{2}\right)}{1}(1)^{-1/2}\left(x^4\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(1)^{-3/2}\left(x^4\right)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(1)^{-5/2}\left(x^4\right)^3$
$+ \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}(1)^{-7/2}\left(x^4\right)^4 + \ldots = 1 + \frac{x^4}{2} - \frac{x^8}{8} + \frac{x^{12}}{16} - \frac{5x^{16}}{128} + \ldots$
$\Rightarrow \int_0^{0.1}\left(1 + \frac{x^4}{2} - \frac{x^8}{8} + \frac{x^{12}}{16} - \frac{5x^{16}}{128} + \ldots\right) dx \approx \left[x + \frac{x^5}{10}\right]_0^{0.1} \approx 0.100001$, $|E| \le \frac{(0.1)^9}{72} \approx 1.39 \times 10^{-11}$

22. $\int_0^1 \left(\frac{1-\cos x}{x^2}\right) dx = \int_0^1\left(\frac{1}{2} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \frac{x^8}{10!} - \ldots\right) dx \approx \left[\frac{x}{2} - \frac{x^3}{3\cdot 4!} + \frac{x^5}{5\cdot 6!} - \frac{x^7}{7\cdot 8!} + \frac{x^9}{9\cdot 10!}\right]_0^1$
≈ 0.4863853764, $|E| \le \frac{1}{11\cdot 12!} \approx 1.9 \times 10^{-10}$

23. $\int_0^1 \cos t^2\, dt = \int_0^1\left(1 - \frac{t^4}{2} + \frac{t^8}{4!} - \frac{t^{12}}{6!} + \ldots\right) dt = \left[t - \frac{t^5}{10} + \frac{t^9}{9\cdot 4!} - \frac{t^{13}}{13\cdot 6!} + \ldots\right]_0^1 \Rightarrow |\text{error}| < \frac{1}{13\cdot 6!} \approx .00011$

24. $\int_0^1 \cos\sqrt{t}\,dt = \int_0^1 \left(1 - \frac{t}{2} + \frac{t^2}{4!} - \frac{t^3}{6!} + \frac{t^4}{8!} - \ldots\right) dt = \left[t - \frac{t^2}{4} + \frac{t^3}{3\cdot 4!} - \frac{t^4}{4\cdot 6!} + \frac{t^5}{5\cdot 8!} - \ldots\right]_0^1$
$\Rightarrow\ |\text{error}| < \frac{1}{5\cdot 8!} \approx 0.000004960$

25. $F(x) = \int_0^x \left(t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \frac{t^{14}}{7!} + \ldots\right) dt = \left[\frac{t^3}{3} - \frac{t^7}{7\cdot 3!} + \frac{t^{11}}{11\cdot 5!} - \frac{t^{15}}{15\cdot 7!} + \ldots\right]_0^x \approx \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!}$
$\Rightarrow\ |\text{error}| < \frac{1}{15\cdot 7!} \approx 0.000013$

26. $F(x) = \int_0^x \left(t^2 - t^4 + \frac{t^6}{2!} - \frac{t^8}{3!} + \frac{t^{10}}{4!} - \frac{t^{12}}{5!} + \ldots\right) dt = \left[\frac{t^3}{3} - \frac{t^5}{5} + \frac{t^7}{7\cdot 2!} - \frac{t^9}{9\cdot 3!} + \frac{t^{11}}{11\cdot 4!} - \frac{t^{13}}{13\cdot 5!} + \ldots\right]_0^x$
$\approx \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7\cdot 2!} - \frac{x^9}{9\cdot 3!} + \frac{x^{11}}{11\cdot 4!}\ \Rightarrow\ |\text{error}| < \frac{1}{13\cdot 5!} \approx 0.00064$

27. (a) $F(x) = \int_0^x \left(t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \ldots\right) dt = \left[\frac{t^2}{2} - \frac{t^4}{12} + \frac{t^6}{30} - \ldots\right]_0^x \approx \frac{x^2}{2} - \frac{x^4}{12}\ \Rightarrow\ |\text{error}| < \frac{(0.5)^6}{30} \approx .00052$
(b) $|\text{error}| < \frac{1}{33\cdot 34} \approx .00089$ when $F(x) \approx \frac{x^2}{2} - \frac{x^4}{3\cdot 4} + \frac{x^6}{5\cdot 6} - \frac{x^8}{7\cdot 8} + \ldots + (-1)^{15}\frac{x^{32}}{31\cdot 32}$

28. (a) $F(x) = \int_0^x \left(1 - \frac{t}{2} + \frac{t^2}{3} - \frac{t^3}{4} + \ldots\right) dt = \left[t - \frac{t^2}{2\cdot 2} + \frac{t^3}{3\cdot 3} - \frac{t^4}{4\cdot 4} + \frac{t^5}{5\cdot 5} - \ldots\right]_0^x \approx x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \frac{x^5}{5^2}$
$\Rightarrow\ |\text{error}| < \frac{(0.5)^6}{6^2} \approx .00043$
(b) $|\text{error}| < \frac{1}{32^2} \approx .00097$ when $F(x) \approx x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots + (-1)^{31}\frac{x^{31}}{31^2}$

29. $\frac{1}{x^2}\left(e^x - (1 + x)\right) = \frac{1}{x^2}\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots\right) - 1 - x\right) = \frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \ldots\ \Rightarrow\ \lim\limits_{x\to 0} \frac{e^x - (1+x)}{x^2}$
$= \lim\limits_{x\to 0}\left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \ldots\right) = \frac{1}{2}$

30. $\frac{1}{x}\left(e^x - e^{-x}\right) = \frac{1}{x}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\right)\right] = \frac{1}{x}\left(2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \frac{2x^7}{7!} + \ldots\right)$
$= 2 + \frac{2x^2}{3!} + \frac{2x^4}{5!} + \frac{2x^6}{7!} + \ldots\ \Rightarrow\ \lim\limits_{x\to 0} \frac{e^x - e^{-x}}{x} = \lim\limits_{x\to\infty}\left(2 + \frac{2x^2}{3!} + \frac{2x^4}{5!} + \frac{2x^6}{7!} + \ldots\right) = 2$

31. $\frac{1}{t^4}\left(1 - \cos t - \frac{t^2}{2}\right) = \frac{1}{t^4}\left[1 - \frac{t^2}{2} - \left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots\right)\right] = -\frac{1}{4!} + \frac{t^2}{6!} - \frac{t^4}{8!} + \ldots\ \Rightarrow\ \lim\limits_{t\to 0} \frac{1 - \cos t - \left(\frac{t^2}{2}\right)}{t^4}$
$= \lim\limits_{t\to 0}\left(-\frac{1}{4!} + \frac{t^2}{6!} - \frac{t^4}{8!} + \ldots\right) = -\frac{1}{24}$

32. $\frac{1}{\theta^5}\left(-\theta + \frac{\theta^3}{6} + \sin\theta\right) = \frac{1}{\theta^5}\left(-\theta + \frac{\theta^3}{6} + \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right) = \frac{1}{5!} - \frac{\theta^2}{7!} + \frac{\theta^4}{9!} - \ldots\ \Rightarrow\ \lim\limits_{\theta\to 0} \frac{\sin\theta - \theta + \left(\frac{\theta^3}{6}\right)}{\theta^5}$
$= \lim\limits_{\theta\to 0}\left(\frac{1}{5!} - \frac{\theta^2}{7!} + \frac{\theta^4}{9!} - \ldots\right) = \frac{1}{120}$

33. $\frac{1}{y^3}\left(y - \tan^{-1} y\right) = \frac{1}{y^3}\left[y - \left(y - \frac{y^3}{3} + \frac{y^5}{5} - \ldots\right)\right] = \frac{1}{3} - \frac{y^2}{5} + \frac{y^4}{7} - \ldots\ \Rightarrow\ \lim\limits_{y\to 0} \frac{y - \tan^{-1} y}{y^3} = \lim\limits_{y\to 0}\left(\frac{1}{3} - \frac{y^2}{5} + \frac{y^4}{7} - \ldots\right)$
$= \frac{1}{3}$

34. $\frac{\tan^{-1} y - \sin y}{y^3 \cos y} = \frac{\left(y - \frac{y^3}{3} + \frac{y^5}{5} - \ldots\right) - \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots\right)}{y^3 \cos y} = \frac{\left(-\frac{y^3}{6} + \frac{23y^5}{5!} - \ldots\right)}{y^3 \cos y} = \frac{\left(-\frac{1}{6} + \frac{23y^2}{5!} - \ldots\right)}{\cos y}$
$\Rightarrow\ \lim\limits_{y\to 0} \frac{\tan^{-1} y - \sin y}{y^3 \cos y} = \lim\limits_{y\to 0} \frac{\left(-\frac{1}{6} + \frac{23y^2}{5!} - \ldots\right)}{\cos y} = -\frac{1}{6}$

35. $x^2\left(-1 + e^{-1/x^2}\right) = x^2\left(-1 + 1 - \frac{1}{x^2} + \frac{1}{2x^4} - \frac{1}{6x^6} + \ldots\right) = -1 + \frac{1}{2x^2} - \frac{1}{6x^4} + \ldots\ \Rightarrow\ \lim\limits_{x\to\infty} x^2\left(e^{-1/x^2} - 1\right)$
$= \lim\limits_{x\to\infty}\left(-1 + \frac{1}{2x^2} - \frac{1}{6x^4} + \ldots\right) = -1$

36. $(x+1)\sin\left(\frac{1}{x+1}\right) = (x+1)\left(\frac{1}{x+1} - \frac{1}{3!(x+1)^3} + \frac{1}{5!(x+1)^5} - \ldots\right) = 1 - \frac{1}{3!(x+1)^2} + \frac{1}{5!(x+1)^4} - \ldots$
$\Rightarrow \lim_{x\to\infty} (x+1)\sin\left(\frac{1}{x+1}\right) = \lim_{x\to\infty}\left(1 - \frac{1}{3!(x+1)^2} + \frac{1}{5!(x+1)^4} - \ldots\right) = 1$

37. $\frac{\ln(1+x^2)}{1-\cos x} = \frac{\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \ldots\right)}{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)} = \frac{\left(1-\frac{x^2}{2}+\frac{x^4}{3}-\ldots\right)}{\left(\frac{1}{2!}-\frac{x^2}{4!}+\ldots\right)} \Rightarrow \lim_{x\to 0} \frac{\ln(1+x^2)}{1-\cos x} = \lim_{x\to 0} \frac{\left(1-\frac{x^2}{2}+\frac{x^4}{3}-\ldots\right)}{\left(\frac{1}{2!}-\frac{x^2}{4!}+\ldots\right)} = 2! = 2$

38. $\frac{x^2-4}{\ln(x-1)} = \frac{(x-2)(x+2)}{\left[(x-2)-\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}-\ldots\right]} = \frac{x+2}{\left[1-\frac{x-2}{2}+\frac{(x-2)^2}{3}-\ldots\right]} \Rightarrow \lim_{x\to 2} \frac{x^2-4}{\ln(x-1)}$
$= \lim_{x\to 2} \frac{x+2}{\left[1-\frac{x-2}{2}+\frac{(x-2)^2}{3}-\ldots\right]} = 4$

39. $\sin 3x^2 = 3x^2 - \frac{9}{2}x^6 + \frac{81}{40}x^{10} - \ldots$ and $1-\cos 2x = 2x^2 - \frac{2}{3}x^4 + \frac{4}{45}x^6 - \ldots \Rightarrow \lim_{x\to 0} \frac{\sin 3x^2}{1-\cos 2x}$
$= \lim_{x\to 0} \frac{3x^2 - \frac{9}{2}x^6 + \frac{81}{40}x^{10} - \ldots}{2x^2 - \frac{2}{3}x^4 + \frac{4}{45}x^6 - \ldots} = \lim_{x\to 0} \frac{3 - \frac{9}{2}x^4 + \frac{81}{40}x^8 - \ldots}{2 - \frac{2}{3}x^2 + \frac{4}{45}x^4 - \ldots} = \frac{3}{2}$

40. $\ln(1+x^3) = x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \ldots$ and $x\sin x^2 = x^3 - \frac{1}{6}x^7 + \frac{1}{120}x^{11} - \frac{1}{5040}x^{15} + \ldots \Rightarrow \lim_{x\to 0} \frac{\ln(1+x^3)}{x\sin x^2}$
$= \lim_{x\to 0} \frac{x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \ldots}{x^3 - \frac{1}{6}x^7 + \frac{1}{120}x^{11} - \frac{1}{5040}x^{15} + \ldots} = \lim_{x\to 0} \frac{1 - \frac{x^3}{2} + \frac{x^6}{3} - \frac{x^9}{4} + \ldots}{1 - \frac{1}{6}x^4 + \frac{1}{120}x^8 - \frac{1}{5040}x^{12} + \ldots} = 1$

41. $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots = e^1 = e$

42. $\left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5 + \ldots = \left(\frac{1}{4}\right)^3\left[1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \ldots\right] = \frac{1}{64}\frac{1}{1-1/4} = \frac{1}{64}\frac{4}{3} = \frac{1}{48}$

43. $1 - \frac{3^2}{4^2 2!} + \frac{3^4}{4^4 4!} - \frac{3^6}{4^6 6!} + \ldots = 1 - \frac{1}{2!}\left(\frac{3}{4}\right)^2 + \frac{1}{4!}\left(\frac{3}{4}\right)^4 - \frac{1}{6!}\left(\frac{3}{4}\right)^6 + \ldots = \cos\left(\frac{3}{4}\right)$

44. $\frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \ldots = \left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 - \frac{1}{4}\left(\frac{1}{2}\right)^4 + \ldots = \ln\left(1+\frac{1}{2}\right) = \ln\left(\frac{3}{2}\right)$

45. $\frac{\pi}{3} - \frac{\pi^3}{3^3 3!} + \frac{\pi^5}{3^5 5!} - \frac{\pi^7}{3^7 7!} + \ldots = \frac{\pi}{3} - \frac{1}{3!}\left(\frac{\pi}{3}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{3}\right)^5 - \frac{1}{7!}\left(\frac{\pi}{3}\right)^7 + \ldots = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

46. $\frac{2}{3} - \frac{2^3}{3^3\cdot 3} + \frac{2^5}{3^5\cdot 5} - \frac{2^7}{3^7\cdot 7} + \ldots = \left(\frac{2}{3}\right) - \frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{5}\left(\frac{2}{3}\right)^5 - \frac{1}{7}\left(\frac{2}{3}\right)^7 + \ldots = \tan^{-1}\left(\frac{2}{3}\right)$

47. $x^3 + x^4 + x^5 + x^6 + \ldots = x^3(1 + x + x^2 + x^3 + \ldots) = x^3\left(\frac{1}{1-x}\right) = \frac{x^3}{1-x}$

48. $1 - \frac{3^2x^2}{2!} + \frac{3^4x^4}{4!} - \frac{3^6x^6}{6!} + \ldots = 1 - \frac{1}{2!}(3x)^2 + \frac{1}{4!}(3x)^4 - \frac{1}{6!}(3x)^6 + \ldots = \cos(3x)$

49. $x^3 - x^5 + x^7 - x^9 + \ldots = x^3\left(1 - x^2 + (x^2)^2 - (x^2)^3 + \ldots\right) = x^3\left(\frac{1}{1+x^2}\right) = \frac{x^3}{1+x^2}$

50. $x^2 - 2x^3 + \frac{2^2x^4}{2!} - \frac{2^3x^5}{3!} + \frac{2^4x^6}{4!} - \ldots = x^2\left(1 - 2x + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} - \ldots\right) = x^2e^{-2x}$

51. $-1 + 2x - 3x^2 + 4x^3 - 5x^4 + \ldots = \frac{d}{dx}(1 - x + x^2 - x^3 + x^4 - x^5 + \ldots) = \frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{-1}{(1+x)^2}$

52. $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \ldots = -\frac{1}{x}\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots\right) = -\frac{1}{x}\ln(1-x) = -\frac{\ln(1-x)}{x}$

53. $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$

54. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n-1}x^n}{n} + \ldots \Rightarrow |\text{error}| = \left|\frac{(-1)^{n-1}x^n}{n}\right| = \frac{1}{n10^n}$ when $x = 0.1$;
$\frac{1}{n10^n} < \frac{1}{10^8} \Rightarrow n10^n > 10^8$ when $n \geq 8 \Rightarrow$ 7 terms

55. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots + \frac{(-1)^{n-1}x^{2n-1}}{2n-1} + \ldots \Rightarrow |\text{error}| = \left|\frac{(-1)^{n-1}x^{2n-1}}{2n-1}\right| = \frac{1}{2n-1}$ when $x = 1$;
$\frac{1}{2n-1} < \frac{1}{10^3} \Rightarrow n > \frac{1001}{2} = 500.5 \Rightarrow$ the first term not used is the 501st $\Rightarrow$ we must use 500 terms

56. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots + \frac{(-1)^{n-1}x^{2n-1}}{2n-1} + \ldots$ and $\lim_{n\to\infty} \left|\frac{x^{2n+1}}{2n+1} \cdot \frac{2n-1}{x^{2n-1}}\right| = x^2 \lim_{n\to\infty} \left|\frac{2n-1}{2n+1}\right| = x^2$
$\Rightarrow \tan^{-1} x$ converges for $|x| < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ which is a convergent series; when $x = 1$
we have $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ which is a convergent series $\Rightarrow$ the series representing $\tan^{-1} x$ diverges for $|x| > 1$

57. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots + \frac{(-1)^{n-1}x^{2n-1}}{2n-1} + \ldots$ and when the series representing $48 \tan^{-1}\left(\frac{1}{18}\right)$ has an error less than $\frac{1}{3} \cdot 10^{-6}$, then the series representing the sum
$48 \tan^{-1}\left(\frac{1}{18}\right) + 32 \tan^{-1}\left(\frac{1}{57}\right) - 20 \tan^{-1}\left(\frac{1}{239}\right)$ also has an error of magnitude less than 10^{-6}; thus
$|\text{error}| = 48 \frac{\left(\frac{1}{18}\right)^{2n-1}}{2n-1} < \frac{1}{3 \cdot 10^6} \Rightarrow n \geq 4$ using a calculator $\Rightarrow$ 4 terms

58. $\ln(\sec x) = \int_0^x \tan t\, dt = \int_0^x \left(t + \frac{t^3}{3} + \frac{2t^5}{15} + \ldots\right) dt \approx \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \ldots$

59. (a) $(1-x^2)^{-1/2} \approx 1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16} \Rightarrow \sin^{-1} x \approx x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112}$; Using the Ratio Test:
$\lim_{n\to\infty} \left|\frac{1\cdot3\cdot5\cdots(2n-1)(2n+1)x^{2n+3}}{2\cdot4\cdot6\cdots(2n)(2n+2)(2n+3)} \cdot \frac{2\cdot4\cdot6\cdots(2n)(2n+1)}{1\cdot3\cdot5\cdots(2n-1)x^{2n+1}}\right| < 1 \Rightarrow x^2 \lim_{n\to\infty} \left|\frac{(2n+1)(2n+1)}{(2n+2)(2n+3)}\right| < 1$
$\Rightarrow |x| < 1 \Rightarrow$ the radius of convergence is 1. See Exercise 69.
(b) $\frac{d}{dx}(\cos^{-1} x) = -(1-x^2)^{-1/2} \Rightarrow \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x \approx \frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112}\right) \approx \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} - \frac{5x^7}{112}$

60. (a) $(1+t^2)^{-1/2} \approx (1)^{-1/2} + \left(-\frac{1}{2}\right)(1)^{-3/2}(t^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1)^{-5/2}(t^2)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(1)^{-7/2}(t^2)^3}{3!}$
$= 1 - \frac{t^2}{2} + \frac{3t^4}{2^2 \cdot 2!} - \frac{3\cdot5t^6}{2^3\cdot3!} \Rightarrow \sinh^{-1} x \approx \int_0^x \left(1 - \frac{t^2}{2} + \frac{3t^4}{8} - \frac{5t^6}{16}\right) dt = x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{5x^7}{112}$
(b) $\sinh^{-1}\left(\frac{1}{4}\right) \approx \frac{1}{4} - \frac{1}{384} + \frac{3}{40{,}960} = 0.24746908$; the error is less than the absolute value of the first unused term, $\frac{5x^7}{112}$, evaluated at $t = \frac{1}{4}$ since the series is alternating $\Rightarrow |\text{error}| < \frac{5\left(\frac{1}{4}\right)^7}{112} \approx 2.725 \times 10^{-6}$

61. $\frac{-1}{1+x} = -\frac{1}{1-(-x)} = -1 + x - x^2 + x^3 - \ldots \Rightarrow \frac{d}{dx}\left(\frac{-1}{1+x}\right) = \frac{1}{1+x^2} = \frac{d}{dx}(-1 + x - x^2 + x^3 - \ldots)$
$= 1 - 2x + 3x^2 - 4x^3 + \ldots$

62. $\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \ldots \Rightarrow \frac{d}{dx}\left(\frac{1}{1-x^2}\right) = \frac{2x}{(1-x^2)^2} = \frac{d}{dx}(1 + x^2 + x^4 + x^6 + \ldots) = 2x + 4x^3 + 6x^5 + \ldots$

63. Wallis' formula gives the approximation $\pi \approx 4\left[\frac{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdots(2n-2)\cdot(2n)}{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdots(2n-1)\cdot(2n-1)}\right]$ to produce the table

n	$\sim\pi$
10	3.221088998
20	3.181104886
30	3.167880758
80	3.151425420
90	3.150331383
93	3.150049112
94	3.149959030
95	3.149870848
100	3.149456425

At $n = 1929$ we obtain the first approximation accurate to 3 decimals: 3.141999845. At $n = 30{,}000$ we still do not obtain accuracy to 4 decimals: 3.141617732, so the convergence to π is very slow. Here is a Maple CAS procedure to produce these approximations:

```
pie :=
   proc(n)
   local i,j;
      a(2) := evalf(8/9);
      for i from 3 to n do a(i) := evalf(2*(2*i−2)*i/(2*i−1)^2*a(i−1)) od;
      [[j,4*a(j)] $ (j = n−5 .. n)]
   end
```

64. (a) $f(x) = 1 + \sum_{k=1}^{\infty}\binom{m}{k}x^k \Rightarrow f'(x) = \sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k-1} \Rightarrow (1+x)\cdot f'(x) = (1+x)\sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k-1}$

$= \sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k-1} + x\cdot\sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k-1} = \sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k-1} + \sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k} = \binom{m}{1}(1)\,x^0 + \sum_{k=2}^{\infty}\binom{m}{k}k\,x^{k-1} + \sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k}$

$= m + \sum_{k=2}^{\infty}\binom{m}{k}k\,x^{k-1} + \sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k}$ Note that: $\sum_{k=2}^{\infty}\binom{m}{k}k\,x^{k-1} = \sum_{k=1}^{\infty}\binom{m}{k+1}(k+1)\,x^{k}$.

Thus, $(1+x)\cdot f'(x) = m + \sum_{k=2}^{\infty}\binom{m}{k}k\,x^{k-1} + \sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k} = m + \sum_{k=1}^{\infty}\binom{m}{k+1}(k+1)\,x^{k} + \sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k}$

$= m + \sum_{k=1}^{\infty}\left[\binom{m}{k+1}(k+1)\,x^k + \binom{m}{k}k\,x^k\right] = m + \sum_{k=1}^{\infty}\left[\left(\binom{m}{k+1}(k+1) + \binom{m}{k}k\right)x^k\right]$.

Note that: $\binom{m}{k+1}(k+1) + \binom{m}{k}k = \frac{m\cdot(m-1)\cdots(m-(k+1)+1)}{(k+1)!}(k+1) + \frac{m\cdot(m-1)\cdots(m-k+1)}{k!}k$

$= \frac{m\cdot(m-1)\cdots(m-k)}{k!} + \frac{m\cdot(m-1)\cdots(m-k+1)}{k!}k = \frac{m\cdot(m-1)\cdots(m-k+1)}{k!}((m-k)+k) = m\frac{m\cdot(m-1)\cdots(m-k+1)}{k!} = m\binom{m}{k}$.

Thus, $(1+x)\cdot f'(x) = m + \sum_{k=1}^{\infty}\left[\left(\binom{m}{k+1}(k+1) + \binom{m}{k}k\right)x^k\right] = m + \sum_{k=1}^{\infty}\left[\left(m\binom{m}{k}\right)x^k\right] = m + m\sum_{k=1}^{\infty}\binom{m}{k}x^k$

$= m\left(1 + \sum_{k=1}^{\infty}\binom{m}{k}x^k\right) = m\cdot f(x) \Rightarrow f'(x) = \frac{m\cdot f(x)}{(1+x)}$ if $-1 < x < 1$.

(b) Let $g(x) = (1+x)^{-m}f(x) \Rightarrow g'(x) = -m(1+x)^{-m-1}f(x) + (1+x)^{-m}f'(x)$

$= -m(1+x)^{-m-1}f(x) + (1+x)^{-m}\cdot\frac{m\cdot f(x)}{(1+x)} = -m(1+x)^{-m-1}f(x) + (1+x)^{-m-1}\cdot m\cdot f(x) = 0.$

(c) $g'(x) = 0 \Rightarrow g(x) = c \Rightarrow (1+x)^{-m}f(x) = c \Rightarrow f(x) = \frac{c}{(1+x)^{-m}} = c(1+x)^m$. Since $f(x) = 1 + \sum_{k=1}^{\infty}\binom{m}{k}x^k$

$\Rightarrow f(0) = 1 + \sum_{k=1}^{\infty}\binom{m}{k}(0)^k = 1 + 0 = 1 \Rightarrow c(1+0)^m = 1 \Rightarrow c = 1 \Rightarrow f(x) = (1+x)^m$.

65. $(1-x^2)^{-1/2} = (1+(-x^2))^{-1/2} = (1)^{-1/2} + \left(-\frac{1}{2}\right)(1)^{-3/2}(-x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1)^{-5/2}(-x^2)^2}{2!}$

$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(1)^{-7/2}(-x^2)^3}{3!} + \ldots = 1 + \frac{x^2}{2} + \frac{1\cdot 3x^4}{2^2\cdot 2!} + \frac{1\cdot 3\cdot 5x^6}{2^3\cdot 3!} + \ldots = 1 + \sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)x^{2n}}{2^n\cdot n!}$

$\Rightarrow \sin^{-1} x = \int_0^x (1-t^2)^{-1/2}\,dt = \int_0^x \left(1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)x^{2n}}{2^n\cdot n!}\right) dt = x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)x^{2n+1}}{2\cdot 4\cdots(2n)(2n+1)}$,
where $|x| < 1$

66. $[\tan^{-1} t]_x^{\infty} = \frac{\pi}{2} - \tan^{-1} x = \int_x^{\infty} \frac{dt}{1+t^2} = \int_x^{\infty} \left[\frac{\left(\frac{1}{t^2}\right)}{1+\left(\frac{1}{t^2}\right)}\right] dt = \int_x^{\infty} \frac{1}{t^2}\left(1 - \frac{1}{t^2} + \frac{1}{t^4} - \frac{1}{t^6} + \ldots\right) dt$
$= \int_x^{\infty}\left(\frac{1}{t^2} - \frac{1}{t^4} + \frac{1}{t^6} - \frac{1}{t^8} + \ldots\right) dt = \lim_{b\to\infty} \left[-\frac{1}{t} + \frac{1}{3t^3} - \frac{1}{5t^5} + \frac{1}{7t^7} - \ldots\right]_x^b = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \ldots$
$\Rightarrow \tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \ldots,\ x > 1;\ [\tan^{-1} t]_{-\infty}^{x} = \tan^{-1} x + \frac{\pi}{2} = \int_{-\infty}^{x} \frac{dt}{1+t^2}$
$= \lim_{b\to-\infty} \left[-\frac{1}{t} + \frac{1}{3t^3} - \frac{1}{5t^5} + \frac{1}{7t^7} - \ldots\right]_b^x = -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \ldots \Rightarrow \tan^{-1} x = -\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \ldots,$
$x < -1$

67. (a) $e^{-i\pi} = \cos(-\pi) + i\sin(-\pi) = -1 + i(0) = -1$
(b) $e^{i\pi/4} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}\right)(1+i)$
(c) $e^{-i\pi/2} = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = 0 + i(-1) = -i$

68. $e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta;$
$e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta \Rightarrow \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2};$
$e^{i\theta} - e^{-i\theta} = \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta) = 2i\sin\theta \Rightarrow \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

69. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \Rightarrow e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots$ and
$e^{-i\theta} = 1 - i\theta + \frac{(-i\theta)^2}{2!} + \frac{(-i\theta)^3}{3!} + \frac{(-i\theta)^4}{4!} + \ldots = 1 - i\theta + \frac{(i\theta)^2}{2!} - \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} - \ldots$
$\Rightarrow \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{\left(1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots\right) + \left(1 - i\theta + \frac{(i\theta)^2}{2!} - \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} - \ldots\right)}{2}$
$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots = \cos\theta;$
$\frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{\left(1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots\right) - \left(1 - i\theta + \frac{(i\theta)^2}{2!} - \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} - \ldots\right)}{2i}$
$= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots = \sin\theta$

70. $e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$
(a) $e^{i\theta} + e^{-i\theta} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) = 2\cos\theta \Rightarrow \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \cosh i\theta$
(b) $e^{i\theta} - e^{-i\theta} = (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta) = 2i\sin\theta \Rightarrow i\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2} = \sinh i\theta$

71. $e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$
$= (1)x + (1)x^2 + \left(-\frac{1}{6} + \frac{1}{2}\right)x^3 + \left(-\frac{1}{6} + \frac{1}{6}\right)x^4 + \left(\frac{1}{120} - \frac{1}{12} + \frac{1}{24}\right)x^5 + \ldots = x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \ldots;$
$e^x \cdot e^{ix} = e^{(1+i)x} = e^x(\cos x + i\sin x) = e^x\cos x + i(e^x\sin x) \Rightarrow e^x\sin x$ is the series of the imaginary part of $e^{(1+i)x}$ which we calculate next; $e^{(1+i)x} = \sum_{n=0}^{\infty} \frac{(x+ix)^n}{n!} = 1 + (x+ix) + \frac{(x+ix)^2}{2!} + \frac{(x+ix)^3}{3!} + \frac{(x+ix)^4}{4!} + \ldots$
$= 1 + x + ix + \frac{1}{2!}(2ix^2) + \frac{1}{3!}(2ix^3 - 2x^3) + \frac{1}{4!}(-4x^4) + \frac{1}{5!}(-4x^5 - 4ix^5) + \frac{1}{6!}(-8ix^6) + \ldots \Rightarrow$ the imaginary part of $e^{(1+i)x}$ is $x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 - \frac{4}{5!}x^5 - \frac{8}{6!}x^6 + \ldots = x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 - \frac{1}{90}x^6 + \ldots$ in agreement with our product calculation. The series for $e^x\sin x$ converges for all values of x.

72. $\frac{d}{dx}\left(e^{(a+ib)}\right) = \frac{d}{dx}[e^{ax}(\cos bx + i\sin bx)] = ae^{ax}(\cos bx + i\sin bx) + e^{ax}(-b\sin bx + bi\cos bx)$
$= ae^{ax}(\cos bx + i\sin bx) + bie^{ax}(\cos bx + i\sin bx) = ae^{(a+ib)x} + ibe^{(a+ib)x} = (a+ib)e^{(a+ib)x}$

73. (a) $e^{i\theta_1}e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)$
$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = e^{i(\theta_1+\theta_2)}$

(b) $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta = (\cos\theta - i\sin\theta)\left(\frac{\cos\theta + i\sin\theta}{\cos\theta + i\sin\theta}\right) = \frac{1}{\cos\theta + i\sin\theta} = \frac{1}{e^{i\theta}}$

74. $\frac{a-bi}{a^2+b^2}e^{(a+bi)x} + C_1 + iC_2 = \left(\frac{a-bi}{a^2+b^2}\right)e^{ax}(\cos bx + i\sin bx) + C_1 + iC_2$
$= \frac{e^{ax}}{a^2+b^2}(a\cos bx + ia\sin bx - ib\cos bx + b\sin bx) + C_1 + iC_2$
$= \frac{e^{ax}}{a^2+b^2}[(a\cos bx + b\sin bx) + (a\sin bx - b\cos bx)i] + C_1 + iC_2$
$= \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + C_1 + \frac{ie^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + iC_2;$
$e^{(a+bi)x} = e^{ax}e^{ibx} = e^{ax}(\cos bx + i\sin bx) = e^{ax}\cos bx + ie^{ax}\sin bx$, so that given
$\int e^{(a+bi)x}\,dx = \frac{a-bi}{a^2+b^2}e^{(a+bi)x} + C_1 + iC_2$ we conclude that $\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + C_1$
and $\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + C_2$

CHAPTER 10 PRACTICE EXERCISES

1. converges to 1, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 + \frac{(-1)^n}{n}\right) = 1$

2. converges to 0, since $0 \le a_n \le \frac{2}{\sqrt{n}}$, $\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty}\frac{2}{\sqrt{n}} = 0$ using the Sandwich Theorem for Sequences

3. converges to -1, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1-2^n}{2^n}\right) = \lim_{n\to\infty}\left(\frac{1}{2^n} - 1\right) = -1$

4. converges to 1, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}[1 + (0.9)^n] = 1 + 0 = 1$

5. diverges, since $\left\{\sin\frac{n\pi}{2}\right\} = \{0, 1, 0, -1, 0, 1, \ldots\}$

6. converges to 0, since $\{\sin n\pi\} = \{0, 0, 0, \ldots\}$

7. converges to 0, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\ln n^2}{n} = 2\lim_{n\to\infty}\frac{\left(\frac{1}{n}\right)}{1} = 0$

8. converges to 0, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\ln(2n+1)}{n} = \lim_{n\to\infty}\frac{\left(\frac{2}{2n+1}\right)}{1} = 0$

9. converges to 1, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n+\ln n}{n}\right) = \lim_{n\to\infty}\frac{1+\left(\frac{1}{n}\right)}{1} = 1$

10. converges to 0, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\ln(2n^3+1)}{n} = \lim_{n\to\infty}\frac{\left(\frac{6n^2}{2n^3+1}\right)}{1} = \lim_{n\to\infty}\frac{12n}{6n^2} = \lim_{n\to\infty}\frac{2}{n} = 0$

11. converges to e^{-5}, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n-5}{n}\right)^n = \lim_{n\to\infty}\left(1 + \frac{(-5)}{n}\right)^n = e^{-5}$ by Theorem 5

12. converges to $\frac{1}{e}$, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{-n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$ by Theorem 5

13. converges to 3, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{3^n}{n}\right)^{1/n} = \lim_{n\to\infty}\frac{3}{n^{1/n}} = \frac{3}{1} = 3$ by Theorem 5

14. converges to 1, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{3}{n}\right)^{1/n} = \lim_{n\to\infty}\frac{3^{1/n}}{n^{1/n}} = \frac{1}{1} = 1$ by Theorem 5

15. converges to ln 2, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n\left(2^{1/n}-1\right) = \lim\limits_{n\to\infty} \frac{2^{1/n}-1}{\left(\frac{1}{n}\right)} = \lim\limits_{n\to\infty} \frac{\left[\frac{\left(-2^{1/n}\ln 2\right)}{n^2}\right]}{\left(\frac{-1}{n^2}\right)} = \lim\limits_{n\to\infty} 2^{1/n}\ln 2$ $= 2^0 \cdot \ln 2 = \ln 2$

16. converges to 1, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \sqrt[n]{2n+1} = \lim\limits_{n\to\infty} \exp\left(\frac{\ln(2n+1)}{n}\right) = \lim\limits_{n\to\infty} \exp\left(\frac{\frac{2}{2n+1}}{1}\right) = e^0 = 1$

17. diverges, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{(n+1)!}{n!} = \lim\limits_{n\to\infty} (n+1) = \infty$

18. converges to 0, since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{(-4)^n}{n!} = 0$ by Theorem 5

19. $\frac{1}{(2n-3)(2n-1)} = \frac{\left(\frac{1}{2}\right)}{2n-3} - \frac{\left(\frac{1}{2}\right)}{2n-1} \Rightarrow s_n = \left[\frac{\left(\frac{1}{2}\right)}{3} - \frac{\left(\frac{1}{2}\right)}{5}\right] + \left[\frac{\left(\frac{1}{2}\right)}{5} - \frac{\left(\frac{1}{2}\right)}{7}\right] + \ldots + \left[\frac{\left(\frac{1}{2}\right)}{2n-3} - \frac{\left(\frac{1}{2}\right)}{2n-1}\right] = \frac{\left(\frac{1}{2}\right)}{3} - \frac{\left(\frac{1}{2}\right)}{2n-1}$ $\Rightarrow \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left[\frac{1}{6} - \frac{\left(\frac{1}{2}\right)}{2n-1}\right] = \frac{1}{6}$

20. $\frac{-2}{n(n+1)} = \frac{-2}{n} + \frac{2}{n+1} \Rightarrow s_n = \left(\frac{-2}{2} + \frac{2}{3}\right) + \left(\frac{-2}{3} + \frac{2}{4}\right) + \ldots + \left(\frac{-2}{n} + \frac{2}{n+1}\right) = -\frac{2}{2} + \frac{2}{n+1} \Rightarrow \lim\limits_{n\to\infty} s_n$ $= \lim\limits_{n\to\infty} \left(-1 + \frac{2}{n+1}\right) = -1$

21. $\frac{9}{(3n-1)(3n+2)} = \frac{3}{3n-1} - \frac{3}{3n+2} \Rightarrow s_n = \left(\frac{3}{2} - \frac{3}{5}\right) + \left(\frac{3}{5} - \frac{3}{8}\right) + \left(\frac{3}{8} - \frac{3}{11}\right) + \ldots + \left(\frac{3}{3n-1} - \frac{3}{3n+2}\right)$ $= \frac{3}{2} - \frac{3}{3n+2} \Rightarrow \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left(\frac{3}{2} - \frac{3}{3n+2}\right) = \frac{3}{2}$

22. $\frac{-8}{(4n-3)(4n+1)} = \frac{-2}{4n-3} + \frac{2}{4n+1} \Rightarrow s_n = \left(\frac{-2}{9} + \frac{2}{13}\right) + \left(\frac{-2}{13} + \frac{2}{17}\right) + \left(\frac{-2}{17} + \frac{2}{21}\right) + \ldots + \left(\frac{-2}{4n-3} + \frac{2}{4n+1}\right)$ $= -\frac{2}{9} + \frac{2}{4n+1} \Rightarrow \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left(-\frac{2}{9} + \frac{2}{4n+1}\right) = -\frac{2}{9}$

23. $\sum\limits_{n=0}^{\infty} e^{-n} = \sum\limits_{n=0}^{\infty} \frac{1}{e^n}$, a convergent geometric series with $r = \frac{1}{e}$ and $a = 1 \Rightarrow$ the sum is $\frac{1}{1-\left(\frac{1}{e}\right)} = \frac{e}{e-1}$

24. $\sum\limits_{n=1}^{\infty} (-1)^n \frac{3}{4^n} = \sum\limits_{n=0}^{\infty} \left(-\frac{3}{4}\right)\left(\frac{-1}{4}\right)^n$ a convergent geometric series with $r = -\frac{1}{4}$ and $a = \frac{-3}{4} \Rightarrow$ the sum is $\frac{\left(-\frac{3}{4}\right)}{1-\left(\frac{-1}{4}\right)} = -\frac{3}{5}$

25. diverges, a p-series with $p = \frac{1}{2}$

26. $\sum\limits_{n=1}^{\infty} \frac{-5}{n} = -5\sum\limits_{n=1}^{\infty} \frac{1}{n}$, diverges since it is a nonzero multiple of the divergent harmonic series

27. Since $f(x) = \frac{1}{x^{1/2}} \Rightarrow f'(x) = -\frac{1}{2x^{3/2}} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow a_{n+1} < a_n$, and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. Since $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, the given series converges conditionally.

28. converges absolutely by the Direct Comparison Test since $\frac{1}{2n^3} < \frac{1}{n^3}$ for $n \geq 1$, which is the nth term of a convergent p-series

29. The given series does not converge absolutely by the Direct Comparison Test since $\frac{1}{\ln(n+1)} > \frac{1}{n+1}$, which is the nth term of a divergent series. Since $f(x) = \frac{1}{\ln(x+1)} \Rightarrow f'(x) = -\frac{1}{(\ln(x+1))^2(x+1)} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow a_{n+1} < a_n$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$, the given series converges conditionally by the Alternating Series Test.

30. $\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \left[-(\ln x)^{-1}\right]_2^b = -\lim_{b\to\infty} \left(\frac{1}{\ln b} - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2} \Rightarrow$ the series converges absolutely by the Integral Test

31. converges absolutely by the Direct Comparison Test since $\frac{\ln n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$, the nth term of a convergent p-series

32. diverges by the Direct Comparison Test for $e^{n^n} > n \Rightarrow \ln\left(e^{n^n}\right) > \ln n \Rightarrow n^n > \ln n \Rightarrow \ln n^n > \ln(\ln n)$ $\Rightarrow n \ln n > \ln(\ln n) \Rightarrow \frac{\ln n}{\ln(\ln n)} > \frac{1}{n}$, the nth term of the divergent harmonic series

33. $\lim_{n\to\infty} \frac{\left(\frac{1}{n\sqrt{n^2+1}}\right)}{\left(\frac{1}{n^2}\right)} = \sqrt{\lim_{n\to\infty} \frac{n^2}{n^2+1}} = \sqrt{1} = 1 \Rightarrow$ converges absolutely by the Limit Comparison Test

34. Since $f(x) = \frac{3x^2}{x^3+1} \Rightarrow f'(x) = \frac{3x(2-x^3)}{(x^3+1)^2} < 0$ when $x \geq 2 \Rightarrow a_{n+1} < a_n$ for $n \geq 2$ and $\lim_{n\to\infty} \frac{3n^2}{n^3+1} = 0$, the series converges by the Alternating Series Test. The series does not converge absolutely: By the Limit Comparison Test, $\lim_{n\to\infty} \frac{\left(\frac{3n^2}{n^3+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{3n^3}{n^3+1} = 3$. Therefore the convergence is conditional.

35. converges absolutely by the Ratio Test since $\lim_{n\to\infty} \left[\frac{n+2}{(n+1)!} \cdot \frac{n!}{n+1}\right] = \lim_{n\to\infty} \frac{n+2}{(n+1)^2} = 0 < 1$

36. diverges since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n(n^2+1)}{2n^2+n-1}$ does not exist

37. converges absolutely by the Ratio Test since $\lim_{n\to\infty} \left[\frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n}\right] = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$

38. converges absolutely by the Root Test since $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n 3^n}{n^n}} = \lim_{n\to\infty} \frac{6}{n} = 0 < 1$

39. converges absolutely by the Limit Comparison Test since $\lim_{n\to\infty} \frac{\left(\frac{1}{n^{3/2}}\right)}{\left(\frac{1}{\sqrt{n(n+1)(n+2)}}\right)} = \sqrt{\lim_{n\to\infty} \frac{n(n+1)(n+2)}{n^3}} = 1$

40. converges absolutely by the Limit Comparison Test since $\lim_{n\to\infty} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n\sqrt{n^2-1}}\right)} = \sqrt{\lim_{n\to\infty} \frac{n^2(n^2-1)}{n^4}} = 1$

41. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x+4)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(x+4)^n}\right| < 1 \Rightarrow \frac{|x+4|}{3} \lim_{n\to\infty} \left(\frac{n}{n+1}\right) < 1 \Rightarrow \frac{|x+4|}{3} < 1$ $\Rightarrow |x+4| < 3 \Rightarrow -3 < x+4 < 3 \Rightarrow -7 < x < -1$; at $x = -7$ we have $\sum_{n=1}^\infty \frac{(-1)^n 3^n}{n3^n} = \sum_{n=1}^\infty \frac{(-1)^n}{n}$, the alternating harmonic series, which converges conditionally; at $x = -1$ we have $\sum_{n=1}^\infty \frac{3^n}{n3^n} = \sum_{n=1}^\infty \frac{1}{n}$, the divergent harmonic series

(a) the radius is 3; the interval of convergence is $-7 \leq x < -1$
(b) the interval of absolute convergence is $-7 < x < -1$
(c) the series converges conditionally at $x = -7$

42. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x-1)^{2n}}{(2n+1)!} \cdot \frac{(2n-1)!}{(x-1)^{2n-2}}\right| < 1 \Rightarrow (x-1)^2 \lim_{n\to\infty} \frac{1}{(2n)(2n+1)} = 0 < 1$, which holds for all x
   (a) the radius is ∞; the series converges for all x
   (b) the series converges absolutely for all x
   (c) there are no values for which the series converges conditionally

43. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(3x-1)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(3x-1)^n}\right| < 1 \Rightarrow |3x-1| \lim_{n\to\infty} \frac{n^2}{(n+1)^2} < 1 \Rightarrow |3x-1| < 1$
   $\Rightarrow -1 < 3x-1 < 1 \Rightarrow 0 < 3x < 2 \Rightarrow 0 < x < \frac{2}{3}$; at $x = 0$ we have $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n^2}$
   $= -\sum_{n=1}^{\infty} \frac{1}{n^2}$, a nonzero constant multiple of a convergent p-series, which is absolutely convergent; at $x = \frac{2}{3}$ we have $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$, which converges absolutely
   (a) the radius is $\frac{1}{3}$; the interval of convergence is $0 \le x \le \frac{2}{3}$
   (b) the interval of absolute convergence is $0 \le x \le \frac{2}{3}$
   (c) there are no values for which the series converges conditionally

44. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{n+2}{2n+3} \cdot \frac{(2x+1)^{n+1}}{2^{n+1}} \cdot \frac{2n+1}{n+1} \cdot \frac{2^n}{(2x+1)^n}\right| < 1 \Rightarrow \frac{|2x+1|}{2} \lim_{n\to\infty} \left|\frac{n+2}{2n+3} \cdot \frac{2n+1}{n+1}\right| < 1$
   $\Rightarrow \frac{|2x+1|}{2}(1) < 1 \Rightarrow |2x+1| < 2 \Rightarrow -2 < 2x+1 < 2 \Rightarrow -3 < 2x < 1 \Rightarrow -\frac{3}{2} < x < \frac{1}{2}$; at $x = -\frac{3}{2}$ we have $\sum_{n=1}^{\infty} \frac{n+1}{2n+1} \cdot \frac{(-2)^n}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{2n+1}$ which diverges by the nth-Term Test for Divergence since $\lim_{n\to\infty} \left(\frac{n+1}{2n+1}\right) = \frac{1}{2} \ne 0$; at $x = \frac{1}{2}$ we have $\sum_{n=1}^{\infty} \frac{n+1}{2n+1} \cdot \frac{2^n}{2^n} = \sum_{n=1}^{\infty} \frac{n+1}{2n+1}$, which diverges by the nth-Term Test
   (a) the radius is 1; the interval of convergence is $-\frac{3}{2} < x < \frac{1}{2}$
   (b) the interval of absolute convergence is $-\frac{3}{2} < x < \frac{1}{2}$
   (c) there are no values for which the series converges conditionally

45. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{x^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \left|\left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right)\right| < 1 \Rightarrow \frac{|x|}{e} \lim_{n\to\infty} \left(\frac{1}{n+1}\right) < 1$
   $\Rightarrow \frac{|x|}{e} \cdot 0 < 1$, which holds for all x
   (a) the radius is ∞; the series converges for all x
   (b) the series converges absolutely for all x
   (c) there are no values for which the series converges conditionally

46. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} < 1 \Rightarrow |x| < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating Series Test; when $x = 1$ we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, a divergent p-series
   (a) the radius is 1; the interval of convergence is $-1 \le x < 1$
   (b) the interval of absolute convergence is $-1 < x < 1$
   (c) the series converges conditionally at $x = -1$

47. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(n+2)x^{2n+1}}{3^{n+1}} \cdot \frac{3^n}{(n+1)x^{2n-1}}\right| < 1 \Rightarrow \frac{x^2}{3} \lim_{n\to\infty} \left(\frac{n+2}{n+1}\right) < 1 \Rightarrow -\sqrt{3} < x < \sqrt{3}$;
   the series $\sum_{n=1}^{\infty} -\frac{n+1}{\sqrt{3}}$ and $\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{3}}$, obtained with $x = \pm\sqrt{3}$, both diverge
   (a) the radius is $\sqrt{3}$; the interval of convergence is $-\sqrt{3} < x < \sqrt{3}$
   (b) the interval of absolute convergence is $-\sqrt{3} < x < \sqrt{3}$
   (c) there are no values for which the series converges conditionally

48. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x-1)x^{2n+3}}{2n+3}\cdot\frac{2n+1}{(x-1)^{2n+1}}\right| < 1 \Rightarrow (x-1)^2\lim_{n\to\infty}\left(\frac{2n+1}{2n+3}\right) < 1 \Rightarrow (x-1)^2(1) < 1$

$\Rightarrow (x-1)^2 < 1 \Rightarrow |x-1| < 1 \Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2$; at $x = 0$ we have $\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^{2n+1}}{2n+1}$

$= \sum_{n=1}^{\infty}\frac{(-1)^{3n+1}}{2n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n+1}$ which converges conditionally by the Alternating Series Test and the fact that $\sum_{n=1}^{\infty}\frac{1}{2n+1}$ diverges; at $x = 2$ we have $\sum_{n=1}^{\infty}\frac{(-1)^n(1)^{2n+1}}{2n+1} = \sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}$, which also converges conditionally

(a) the radius is 1; the interval of convergence is $0 \le x \le 2$
(b) the interval of absolute convergence is $0 < x < 2$
(c) the series converges conditionally at $x = 0$ and $x = 2$

49. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{\operatorname{csch}(n+1)x^{n+1}}{\operatorname{csch}(n)x^n}\right| < 1 \Rightarrow |x|\lim_{n\to\infty}\left|\frac{\left(\frac{2}{e^{n+1}-e^{-n-1}}\right)}{\left(\frac{2}{e^n-e^{-n}}\right)}\right| < 1$

$\Rightarrow |x|\lim_{n\to\infty}\left|\frac{e^{-1}-e^{-2n-1}}{1-e^{-2n-2}}\right| < 1 \Rightarrow \frac{|x|}{e} < 1 \Rightarrow -e < x < e$; the series $\sum_{n=1}^{\infty}(\pm e)^n \operatorname{csch} n$, obtained with $x = \pm e$, both diverge since $\lim_{n\to\infty}(\pm e)^n \operatorname{csch} n \ne 0$

(a) the radius is e; the interval of convergence is $-e < x < e$
(b) the interval of absolute convergence is $-e < x < e$
(c) there are no values for which the series converges conditionally

50. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{x^{n+1}\coth(n+1)}{x^n\coth(n)}\right| < 1 \Rightarrow |x|\lim_{n\to\infty}\left|\frac{1+e^{-2n-2}}{1-e^{-2n-2}}\cdot\frac{1-e^{-2n}}{1+e^{-2n}}\right| < 1 \Rightarrow |x| < 1$

$\Rightarrow -1 < x < 1$; the series $\sum_{n=1}^{\infty}(\pm 1)^n \coth n$, obtained with $x = \pm 1$, both diverge since $\lim_{n\to\infty}(\pm 1)^n \coth n \ne 0$

(a) the radius is 1; the interval of convergence is $-1 < x < 1$
(b) the interval of absolute convergence is $-1 < x < 1$
(c) there are no values for which the series converges conditionally

51. The given series has the form $1 - x + x^2 - x^3 + \ldots + (-x)^n + \ldots = \frac{1}{1+x}$, where $x = \frac{1}{4}$; the sum is $\frac{1}{1+\left(\frac{1}{4}\right)} = \frac{4}{5}$

52. The given series has the form $x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots = \ln(1+x)$, where $x = \frac{2}{3}$; the sum is $\ln\left(\frac{5}{3}\right) \approx 0.510825624$

53. The given series has the form $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots = \sin x$, where $x = \pi$; the sum is $\sin\pi = 0$

54. The given series has the form $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots = \cos x$, where $x = \frac{\pi}{3}$; the sum is $\cos\frac{\pi}{3} = \frac{1}{2}$

55. The given series has the form $1 + x + \frac{x^2}{2!} + \frac{x^2}{3!} + \ldots + \frac{x^n}{n!} + \ldots = e^x$, where $x = \ln 2$; the sum is $e^{\ln(2)} = 2$

56. The given series has the form $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^n\frac{x^{2n-1}}{(2n-1)} + \ldots = \tan^{-1}x$, where $x = \frac{1}{\sqrt{3}}$; the sum is $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

57. Consider $\frac{1}{1-2x}$ as the sum of a convergent geometric series with $a = 1$ and $r = 2x \Rightarrow \frac{1}{1-2x}$

$= 1 + (2x) + (2x)^2 + (2x)^3 + \ldots = \sum_{n=0}^{\infty}(2x)^n = \sum_{n=0}^{\infty}2^n x^n$ where $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$

58. Consider $\frac{1}{1+x^3}$ as the sum of a convergent geometric series with $a = 1$ and $r = -x^3 \Rightarrow \frac{1}{1+x^3} = \frac{1}{1-(-x^3)}$
$= 1 + (-x^3) + (-x^3)^2 + (-x^3)^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n x^{3n}$ where $|-x^3| < 1 \Rightarrow |x^3| < 1 \Rightarrow |x| < 1$

59. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin \pi x = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1} x^{2n+1}}{(2n+1)!}$

60. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin \frac{2x}{3} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{2x}{3}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{3^{2n+1}(2n+1)!}$

61. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos\left(x^{5/3}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^{5/3}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{10n/3}}{(2n)!}$

62. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos\left(\frac{x^3}{\sqrt{5}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x^3}{\sqrt{5}}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{5^n (2n)!}$

63. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{(\pi x/2)} = \sum_{n=0}^{\infty} \frac{\left(\frac{\pi x}{2}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{\pi^n x^n}{2^n n!}$

64. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{\left(-x^2\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

65. $f(x) = \sqrt{3+x^2} = (3+x^2)^{1/2} \Rightarrow f'(x) = x\,(3+x^2)^{-1/2} \Rightarrow f''(x) = -x^2\,(3+x^2)^{-3/2} + (3+x^2)^{-1/2}$
$\Rightarrow f'''(x) = 3x^3\,(3+x^2)^{-5/2} - 3x\,(3+x^2)^{-3/2}$; $f(-1) = 2$, $f'(-1) = -\frac{1}{2}$, $f''(-1) = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}$,
$f'''(-1) = -\frac{3}{32} + \frac{3}{8} = \frac{9}{32} \Rightarrow \sqrt{3+x^2} = 2 - \frac{(x+1)}{2\cdot 1!} + \frac{3(x+1)^2}{2^3\cdot 2!} + \frac{9(x+1)^3}{2^5\cdot 3!} + \ldots$

66. $f(x) = \frac{1}{1-x} = (1-x)^{-1} \Rightarrow f'(x) = (1-x)^{-2} \Rightarrow f''(x) = 2(1-x)^{-3} \Rightarrow f'''(x) = 6(1-x)^{-4}$; $f(2) = -1$, $f'(2) = 1$,
$f''(2) = -2$, $f'''(2) = 6 \Rightarrow \frac{1}{1-x} = -1 + (x-2) - (x-2)^2 + (x-2)^3 - \ldots$

67. $f(x) = \frac{1}{x+1} = (x+1)^{-1} \Rightarrow f'(x) = -(x+1)^{-2} \Rightarrow f''(x) = 2(x+1)^{-3} \Rightarrow f'''(x) = -6(x+1)^{-4}$; $f(3) = \frac{1}{4}$,
$f'(3) = -\frac{1}{4^2}$, $f''(3) = \frac{2}{4^3}$, $f'''(2) = \frac{-6}{4^4} \Rightarrow \frac{1}{x+1} = \frac{1}{4} - \frac{1}{4^2}(x-3) + \frac{1}{4^3}(x-3)^2 - \frac{1}{4^4}(x-3)^3 + \ldots$

68. $f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = -x^{-2} \Rightarrow f''(x) = 2x^{-3} \Rightarrow f'''(x) = -6x^{-4}$; $f(a) = \frac{1}{a}$, $f'(a) = -\frac{1}{a^2}$, $f''(a) = \frac{2}{a^3}$,
$f'''(a) = \frac{-6}{a^4} \Rightarrow \frac{1}{x} = \frac{1}{a} - \frac{1}{a^2}(x-a) + \frac{1}{a^3}(x-a)^2 - \frac{1}{a^4}(x-a)^3 + \ldots$

69. $\int_0^{1/2} \exp\left(-x^3\right) dx = \int_0^{1/2} \left(1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \frac{x^{12}}{4!} + \ldots\right) dx = \left[x - \frac{x^4}{4} + \frac{x^7}{7\cdot 2!} - \frac{x^{10}}{10\cdot 3!} + \frac{x^{13}}{13\cdot 4!} - \ldots\right]_0^{1/2}$
$\approx \frac{1}{2} - \frac{1}{2^4\cdot 4} + \frac{1}{2^7\cdot 7\cdot 2!} - \frac{1}{2^{10}\cdot 10\cdot 3!} + \frac{1}{2^{13}\cdot 13\cdot 4!} - \frac{1}{2^{16}\cdot 16\cdot 5!} \approx 0.484917143$

70. $\int_0^1 x \sin\left(x^3\right) dx = \int_0^1 x \left(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \frac{x^{27}}{9!} + \ldots\right) dx = \int_0^1 \left(x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \frac{x^{22}}{7!} + \frac{x^{28}}{9!} - \ldots\right) dx$
$= \left[\frac{x^5}{5} - \frac{x^{11}}{11\cdot 3!} + \frac{x^{17}}{17\cdot 5!} - \frac{x^{23}}{23\cdot 7!} + \frac{x^{29}}{29\cdot 9!} - \ldots\right]_0^1 \approx 0.185330149$

71. $\int_1^{1/2} \frac{\tan^{-1} x}{x} dx = \int_1^{1/2} \left(1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \frac{x^8}{9} - \frac{x^{10}}{11} + \ldots\right) dx = \left[x - \frac{x^3}{9} + \frac{x^5}{25} - \frac{x^7}{49} + \frac{x^9}{81} - \frac{x^{11}}{121} + \ldots\right]_0^{1/2}$
$\approx \frac{1}{2} - \frac{1}{9\cdot 2^3} + \frac{1}{5^2\cdot 2^5} - \frac{1}{7^2\cdot 2^7} + \frac{1}{9^2\cdot 2^9} - \frac{1}{11^2\cdot 2^{11}} + \frac{1}{13^2\cdot 2^{13}} - \frac{1}{15^2\cdot 2^{15}} + \frac{1}{17^2\cdot 2^{17}} - \frac{1}{19^2\cdot 2^{19}} + \frac{1}{21^2\cdot 2^{21}} \approx 0.4872223583$

72. $\int_0^{1/64} \frac{\tan^{-1} x}{\sqrt{x}}\,dx = \int_0^{1/64} \frac{1}{\sqrt{x}}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right)dx = \int_0^{1/64}\left(x^{1/2} - \frac{1}{3} x^{5/2} + \frac{1}{5} x^{9/2} - \frac{1}{7} x^{13/2} + \dots\right)dx$
$= \left[\frac{2}{3} x^{3/2} - \frac{2}{21}x^{7/2} + \frac{2}{55}x^{11/2} - \frac{2}{105}x^{15/2} + \dots\right]_0^{1/64} = \left(\frac{2}{3\cdot 8^3} - \frac{2}{21\cdot 8^7} + \frac{2}{55\cdot 8^{11}} - \frac{2}{105\cdot 8^{15}} + \dots\right) \approx 0.0013020379$

73. $\lim\limits_{x\to 0} \frac{7\sin x}{e^{2x}-1} = \lim\limits_{x\to 0} \frac{7\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)}{\left(2x + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} + \dots\right)} = \lim\limits_{x\to 0} \frac{7\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right)}{\left(2 + \frac{2^2x}{2!} + \frac{2^3x^2}{3!} + \dots\right)} = \frac{7}{2}$

74. $\lim\limits_{\theta\to 0} \frac{e^\theta - e^{-\theta} - 2\theta}{\theta - \sin\theta} = \lim\limits_{\theta\to 0} \frac{\left(1+\theta+\frac{\theta^2}{2!}+\frac{\theta^3}{3!}+\dots\right) - \left(1-\theta+\frac{\theta^2}{2!}-\frac{\theta^3}{3!}+\dots\right) - 2\theta}{\theta - \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)} = \lim\limits_{\theta\to 0} \frac{2\left(\frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right)}{\left(\frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \dots\right)}$
$= \lim\limits_{\theta\to 0} \frac{2\left(\frac{1}{3!} + \frac{\theta^2}{5!} + \dots\right)}{\left(\frac{1}{3!} - \frac{\theta^2}{5!} + \dots\right)} = 2$

75. $\lim\limits_{t\to 0} \left(\frac{1}{2-2\cos t} - \frac{1}{t^2}\right) = \lim\limits_{t\to 0} \frac{t^2 - 2 + 2\cos t}{2t^2(1-\cos t)} = \lim\limits_{t\to 0} \frac{t^2 - 2 + 2\left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \dots\right)}{2t^2\left(1 - 1 + \frac{t^2}{2} - \frac{t^4}{4!} + \dots\right)} = \lim\limits_{t\to 0} \frac{2\left(\frac{t^4}{4!} - \frac{t^6}{6!} + \dots\right)}{\left(t^4 - \frac{2t^6}{4!} + \dots\right)}$
$= \lim\limits_{t\to 0} \frac{2\left(\frac{1}{4!} - \frac{t^2}{6!} + \dots\right)}{\left(1 - \frac{2t^2}{4!} + \dots\right)} = \frac{1}{12}$

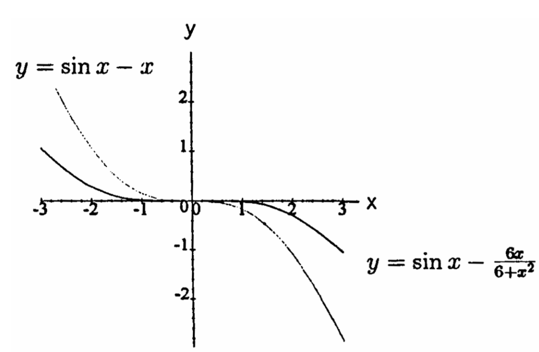
76. $\lim\limits_{h\to 0} \frac{\left(\frac{\sin h}{h}\right) - \cos h}{h^2} = \lim\limits_{h\to 0} \frac{\left(1 - \frac{h^2}{3!} + \frac{h^4}{5!} - \dots\right) - \left(1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \dots\right)}{h^2}$
$= \lim\limits_{h\to 0} \frac{\left(\frac{h^2}{2!} - \frac{h^2}{3!} + \frac{h^4}{5!} - \frac{h^4}{4!} + \frac{h^6}{6!} - \frac{h^6}{7!} + \dots\right)}{h^2} = \lim\limits_{h\to 0} \left(\frac{1}{2!} - \frac{1}{3!} + \frac{h^2}{5!} - \frac{h^2}{4!} + \frac{h^4}{6!} - \frac{h^4}{7!} + \dots\right) = \frac{1}{3}$

77. $\lim\limits_{z\to 0} \frac{1-\cos^2 z}{\ln(1-z) + \sin z} = \lim\limits_{z\to 0} \frac{1 - \left(1 - z^2 + \frac{z^4}{3} - \dots\right)}{\left(-z - \frac{z^2}{2} - \frac{z^3}{3} - \dots\right) + \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)} = \lim\limits_{z\to 0} \frac{\left(z^2 - \frac{z^4}{3} + \dots\right)}{\left(-\frac{z^2}{2} - \frac{2z^3}{3} - \frac{z^4}{4} - \dots\right)}$
$= \lim\limits_{z\to 0} \frac{\left(1 - \frac{z^2}{3} + \dots\right)}{\left(-\frac{1}{2} - \frac{2z}{3} - \frac{z^2}{4} - \dots\right)} = -2$

78. $\lim\limits_{y\to 0} \frac{y^2}{\cos y - \cosh y} = \lim\limits_{y\to 0} \frac{y^2}{\left(1 - \frac{y^2}{2} + \frac{y^4}{4!} - \frac{y^6}{6!} + \dots\right) - \left(1 + \frac{y^2}{2!} + \frac{y^4}{4!} + \frac{y^6}{6!} + \dots\right)} = \lim\limits_{y\to 0} \frac{y^2}{\left(-\frac{2y^2}{2} - \frac{2y^6}{6!} - \dots\right)}$
$= \lim\limits_{y\to 0} \frac{1}{\left(-1 - \frac{2y^4}{6!} - \dots\right)} = -1$

79. $\lim\limits_{x\to 0} \left(\frac{\sin 3x}{x^3} + \frac{r}{x^2} + s\right) = \lim\limits_{x\to 0} \left[\frac{\left(3x - \frac{(3x)^3}{6} + \frac{(3x)^5}{120} - \dots\right)}{x^3} + \frac{r}{x^2} + s\right] = \lim\limits_{x\to 0} \left(\frac{3}{x^2} - \frac{9}{2} + \frac{81x^2}{40} + \dots + \frac{r}{x^2} + s\right) = 0$
$\Rightarrow \frac{r}{x^2} + \frac{3}{x^2} = 0$ and $s - \frac{9}{2} = 0 \Rightarrow r = -3$ and $s = \frac{9}{2}$

80. The approximation $\sin x \approx \frac{6x}{6+x^2}$ is better than $\sin x \approx x$.



81. $\lim\limits_{n\to\infty}\left|\frac{2\cdot5\cdot8\cdots(3n-1)(3n+2)x^{n+1}}{2\cdot4\cdot6\cdots(2n)(2n+2)}\cdot\frac{2\cdot4\cdot6\cdots(2n)}{2\cdot5\cdot8\cdots(3n-1)x^n}\right|<1\Rightarrow|x|\lim\limits_{n\to\infty}\left|\frac{3n+2}{2n+2}\right|<1\Rightarrow|x|<\frac{2}{3}$
$\Rightarrow$ the radius of convergence is $\frac{2}{3}$

82. $\lim\limits_{n\to\infty}\left|\frac{3\cdot5\cdot7\cdots(2n+1)(2n+3)(x-1)^{n+1}}{4\cdot9\cdot14\cdots(5n-1)(5n+4)}\cdot\frac{4\cdot9\cdot14\cdots(5n-1)}{3\cdot5\cdot7\cdots(2n+1)x^n}\right|<1\Rightarrow|x|\lim\limits_{n\to\infty}\left|\frac{2n+3}{5n+4}\right|<1\Rightarrow|x|<\frac{5}{2}$
$\Rightarrow$ the radius of convergence is $\frac{5}{2}$

83. $\sum\limits_{k=2}^{n}\ln\left(1-\frac{1}{k^2}\right)=\sum\limits_{k=2}^{n}\left[\ln\left(1+\frac{1}{k}\right)+\ln\left(1-\frac{1}{k}\right)\right]=\sum\limits_{k=2}^{n}\left[\ln(k+1)-\ln k+\ln(k-1)-\ln k\right]$
$=[\ln 3-\ln 2+\ln 1-\ln 2]+[\ln 4-\ln 3+\ln 2-\ln 3]+[\ln 5-\ln 4+\ln 3-\ln 4]+[\ln 6-\ln 5+\ln 4-\ln 5]$
$+\ldots+[\ln(n+1)-\ln n+\ln(n-1)-\ln n]=[\ln 1-\ln 2]+[\ln(n+1)-\ln n]$ after cancellation
$\Rightarrow\sum\limits_{k=2}^{n}\ln\left(1-\frac{1}{k^2}\right)=\ln\left(\frac{n+1}{2n}\right)\Rightarrow\sum\limits_{k=2}^{\infty}\ln\left(1-\frac{1}{k^2}\right)=\lim\limits_{n\to\infty}\ln\left(\frac{n+1}{2n}\right)=\ln\frac{1}{2}$ is the sum

84. $\sum\limits_{k=2}^{n}\frac{1}{k^2-1}=\frac{1}{2}\sum\limits_{k=2}^{n}\left(\frac{1}{k-1}-\frac{1}{k+1}\right)=\frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\ldots+\left(\frac{1}{n-2}-\frac{1}{n}\right)\right.$
$\left.+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)\right]=\frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right)=\frac{1}{2}\left(\frac{3}{2}-\frac{1}{n}-\frac{1}{n+1}\right)=\frac{1}{2}\left[\frac{3n(n+1)-2(n+1)-2n}{2n(n+1)}\right]=\frac{3n^2-n-2}{4n(n+1)}$
$\Rightarrow\sum\limits_{k=2}^{\infty}\frac{1}{k^2-1}=\lim\limits_{n\to\infty}\frac{1}{2}\left(\frac{3}{2}-\frac{1}{n}-\frac{1}{n+1}\right)=\frac{3}{4}$

85. (a) $\lim\limits_{n\to\infty}\left|\frac{1\cdot4\cdot7\cdots(3n-2)(3n+1)x^{3n+3}}{(3n+3)!}\cdot\frac{(3n)!}{1\cdot4\cdot7\cdots(3n-2)x^{3n}}\right|<1\Rightarrow|x^3|\lim\limits_{n\to\infty}\frac{(3n+1)}{(3n+1)(3n+2)(3n+3)}$
$=|x^3|\cdot 0<1\Rightarrow$ the radius of convergence is ∞

(b) $y=1+\sum\limits_{n=1}^{\infty}\frac{1\cdot4\cdot7\cdots(3n-2)}{(3n)!}x^{3n}\Rightarrow\frac{dy}{dx}=\sum\limits_{n=1}^{\infty}\frac{1\cdot4\cdot7\cdots(3n-2)}{(3n-1)!}x^{3n-1}$
$\Rightarrow\frac{d^2y}{dx^2}=\sum\limits_{n=1}^{\infty}\frac{1\cdot4\cdot7\cdots(3n-2)}{(3n-2)!}x^{3n-2}=x+\sum\limits_{n=2}^{\infty}\frac{1\cdot4\cdot7\cdots(3n-5)}{(3n-3)!}x^{3n-2}$
$=x\left(1+\sum\limits_{n=1}^{\infty}\frac{1\cdot4\cdot7\cdots(3n-2)}{(3n)!}x^{3n}\right)=xy+0\Rightarrow a=1$ and $b=0$

86. (a) $\frac{x^2}{1+x}=\frac{x^2}{1-(-x)}=x^2+x^2(-x)+x^2(-x)^2+x^2(-x)^3+\ldots=x^2-x^3+x^4-x^5+\ldots=\sum\limits_{n=2}^{\infty}(-1)^n x^n$ which converges absolutely for $|x|<1$

(b) $x=1\Rightarrow\sum\limits_{n=2}^{\infty}(-1)^n x^n=\sum\limits_{n=2}^{\infty}(-1)^n$ which diverges

87. Yes, the series $\sum\limits_{n=1}^{\infty}a_nb_n$ converges as we now show. Since $\sum\limits_{n=1}^{\infty}a_n$ converges it follows that $a_n\to 0\Rightarrow a_n<1$ for $n>$ some index $N\Rightarrow a_nb_n<b_n$ for $n>N\Rightarrow\sum\limits_{n=1}^{\infty}a_nb_n$ converges by the Direct Comparison Test with $\sum\limits_{n=1}^{\infty}b_n$

88. No, the series $\sum\limits_{n=1}^{\infty}a_nb_n$ might diverge (as it would if a_n and b_n both equaled n) or it might converge (as it would if a_n and b_n both equaled $\frac{1}{n}$).

89. $\sum\limits_{n=1}^{\infty}(x_{n+1}-x_n)=\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}(x_{k+1}-x_k)=\lim\limits_{n\to\infty}(x_{n+1}-x_1)=\lim\limits_{n\to\infty}(x_{n+1})-x_1\Rightarrow$ both the series and sequence must either converge or diverge.

90. It converges by the Limit Comparison Test since $\lim\limits_{n\to\infty} \frac{\left(\frac{a_n}{1+a_n}\right)}{a_n} = \lim\limits_{n\to\infty} \frac{1}{1+a_n} = 1$ because $\sum\limits_{n=1}^{\infty} a_n$ converges and so $a_n \to 0$.

91. $\sum\limits_{n=1}^{\infty} \frac{a_n}{n} = a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \frac{a_4}{4} + \ldots \ge a_1 + \left(\frac{1}{2}\right) a_2 + \left(\frac{1}{3} + \frac{1}{4}\right) a_4 + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) a_8$
$+ \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \ldots + \frac{1}{16}\right) a_{16} + \ldots \ge \frac{1}{2}(a_2 + a_4 + a_8 + a_{16} + \ldots)$ which is a divergent series

92. $a_n = \frac{1}{\ln n}$ for $n \ge 2 \Rightarrow a_2 \ge a_3 \ge a_4 \ge \ldots$, and $\frac{1}{\ln 2} + \frac{1}{\ln 4} + \frac{1}{\ln 8} + \ldots = \frac{1}{\ln 2} + \frac{1}{2 \ln 2} + \frac{1}{3 \ln 2} + \ldots$
$= \frac{1}{\ln 2}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots\right)$ which diverges so that $1 + \sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by the Integral Test.

CHAPTER 10 ADDITIONAL AND ADVANCED EXERCISES

1. converges since $\frac{1}{(3n-2)^{(2n+1)/2}} < \frac{1}{(3n-2)^{3/2}}$ and $\sum\limits_{n=1}^{\infty} \frac{1}{(3n-2)^{3/2}}$ converges by the Limit Comparison Test:
$\lim\limits_{n\to\infty} \frac{\left(\frac{1}{n^{3/2}}\right)}{\left(\frac{1}{(3n-2)^{3/2}}\right)} = \lim\limits_{n\to\infty} \left(\frac{3n-2}{n}\right)^{3/2} = 3^{3/2}$

2. converges by the Integral Test: $\int_1^{\infty} (\tan^{-1} x)^2 \frac{dx}{x^2+1} = \lim\limits_{b\to\infty} \left[\frac{(\tan^{-1} x)^3}{3}\right]_1^b = \lim\limits_{b\to\infty} \left[\frac{(\tan^{-1} b)^3}{3} - \frac{\pi^3}{192}\right]$
$= \left(\frac{\pi^3}{24} - \frac{\pi^3}{192}\right) = \frac{7\pi^3}{192}$

3. diverges by the nth-Term Test since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (-1)^n \tanh n = \lim\limits_{b\to\infty} (-1)^n \left(\frac{1-e^{-2n}}{1+e^{-2n}}\right) = \lim\limits_{n\to\infty} (-1)^n$ does not exist

4. converges by the Direct Comparison Test: $n! < n^n \Rightarrow \ln(n!) < n \ln(n) \Rightarrow \frac{\ln(n!)}{\ln(n)} < n$
$\Rightarrow \log_n(n!) < n \Rightarrow \frac{\log_n(n!)}{n^3} < \frac{1}{n^2}$, which is the nth-term of a convergent p-series

5. converges by the Direct Comparison Test: $a_1 = 1 = \frac{12}{(1)(3)(2)^2}$, $a_2 = \frac{1\cdot 2}{3\cdot 4} = \frac{12}{(2)(4)(3)^2}$, $a_3 = \left(\frac{2\cdot 3}{4\cdot 5}\right)\left(\frac{1\cdot 2}{3\cdot 4}\right)$
$= \frac{12}{(3)(5)(4)^2}$, $a_4 = \left(\frac{3\cdot 4}{5\cdot 6}\right)\left(\frac{2\cdot 3}{4\cdot 5}\right)\left(\frac{1\cdot 2}{3\cdot 4}\right) = \frac{12}{(4)(6)(5)^2}, \ldots \Rightarrow 1 + \sum\limits_{n=1}^{\infty} \frac{12}{(n+1)(n+3)(n+2)^2}$ represents the given series and $\frac{12}{(n+1)(n+3)(n+2)^2} < \frac{12}{n^4}$, which is the nth-term of a convergent p-series

6. converges by the Ratio Test: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{n}{(n-1)(n+1)} = 0 < 1$

7. diverges by the nth-Term Test since if $a_n \to L$ as $n \to \infty$, then $L = \frac{1}{1+L} \Rightarrow L^2 + L - 1 = 0 \Rightarrow L = \frac{-1 \pm \sqrt{5}}{2} \ne 0$

8. Split the given series into $\sum\limits_{n=1}^{\infty} \frac{1}{3^{2n+1}}$ and $\sum\limits_{n=1}^{\infty} \frac{2n}{3^{2n}}$; the first subseries is a convergent geometric series and the second converges by the Root Test: $\lim\limits_{n\to\infty} \sqrt[n]{\frac{2n}{3^{2n}}} = \lim\limits_{n\to\infty} \frac{\sqrt[n]{2}\,\sqrt[n]{n}}{9} = \frac{1\cdot 1}{9} = \frac{1}{9} < 1$

9. $f(x) = \cos x$ with $a = \frac{\pi}{3} \Rightarrow f\left(\frac{\pi}{3}\right) = 0.5$, $f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$, $f''\left(\frac{\pi}{3}\right) = -0.5$, $f'''\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $f^{(4)}\left(\frac{\pi}{3}\right) = 0.5$;
$\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + \ldots$

10. $f(x) = \sin x$ with $a = 2\pi \Rightarrow f(2\pi) = 0, f'(2\pi) = 1, f''(2\pi) = 0, f'''(2\pi) = -1, f^{(4)}(2\pi) = 0, f^{(5)}(2\pi) = 1,$
$f^{(6)}(2\pi) = 0, f^{(7)}(2\pi) = -1$; $\sin x = (x-2\pi) - \frac{(x-2\pi)^3}{3!} + \frac{(x-2\pi)^5}{5!} - \frac{(x-2\pi)^7}{7!} + \ldots$

11. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ with $a = 0$

12. $f(x) = \ln x$ with $a = 1 \Rightarrow f(1) = 0, f'(1) = 1, f''(1) = -1, f'''(1) = 2, f^{(4)}(1) = -6$;
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$

13. $f(x) = \cos x$ with $a = 22\pi \Rightarrow f(22\pi) = 1, f'(22\pi) = 0, f''(22\pi) = -1, f'''(22\pi) = 0, f^{(4)}(22\pi) = 1,$
$f^{(5)}(22\pi) = 0, f^{(6)}(22\pi) = -1$; $\cos x = 1 - \frac{1}{2}(x-22\pi)^2 + \frac{1}{4!}(x-22\pi)^4 - \frac{1}{6!}(x-22\pi)^6 + \ldots$

14. $f(x) = \tan^{-1} x$ with $a = 1 \Rightarrow f(1) = \frac{\pi}{4}, f'(1) = \frac{1}{2}, f''(1) = -\frac{1}{2}, f'''(1) = \frac{1}{2}$;
$\tan^{-1} x = \frac{\pi}{4} + \frac{(x-1)}{2} - \frac{(x-1)^2}{4} + \frac{(x-1)^3}{12} + \ldots$

15. Yes, the sequence converges: $c_n = (a^n + b^n)^{1/n} \Rightarrow c_n = b\left(\left(\frac{a}{b}\right)^n + 1\right)^{1/n} \Rightarrow \lim_{n\to\infty} c_n = \ln b + \lim_{n\to\infty} \frac{\ln\left(\left(\frac{a}{b}\right)^n + 1\right)}{n}$
$= \ln b + \lim_{n\to\infty} \frac{\left(\frac{a}{b}\right)^n \ln\left(\frac{a}{b}\right)}{\left(\frac{a}{b}\right)^n + 1} = \ln b + \frac{0 \cdot \ln\left(\frac{a}{b}\right)}{0+1} = \ln b$ since $0 < a < b$. Thus, $\lim_{n\to\infty} c_n = e^{\ln b} = b$.

16. $1 + \frac{2}{10} + \frac{3}{10^2} + \frac{7}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \frac{7}{10^6} + \ldots = 1 + \sum_{n=1}^{\infty} \frac{2}{10^{3n-2}} + \sum_{n=1}^{\infty} \frac{3}{10^{3n-1}} + \sum_{n=1}^{\infty} \frac{7}{10^{3n}}$
$= 1 + \sum_{n=0}^{\infty} \frac{2}{10^{3n+1}} + \sum_{n=0}^{\infty} \frac{3}{10^{3n+2}} + \sum_{n=0}^{\infty} \frac{7}{10^{3n+3}} = 1 + \frac{\left(\frac{2}{10}\right)}{1-\left(\frac{1}{10}\right)^3} + \frac{\left(\frac{3}{10^2}\right)}{1-\left(\frac{1}{10}\right)^3} + \frac{\left(\frac{7}{10^3}\right)}{1-\left(\frac{1}{10}\right)^3}$
$= 1 + \frac{200}{999} + \frac{30}{999} + \frac{7}{999} = \frac{999+237}{999} = \frac{412}{333}$

17. $s_n = \sum_{k=0}^{n-1} \int_k^{k+1} \frac{dx}{1+x^2} \Rightarrow s_n = \int_0^1 \frac{dx}{1+x^2} + \int_1^2 \frac{dx}{1+x^2} + \ldots + \int_{n-1}^{n} \frac{dx}{1+x^2} \Rightarrow s_n = \int_0^n \frac{dx}{1+x^2}$
$\Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} (\tan^{-1} n - \tan^{-1} 0) = \frac{\pi}{2}$

18. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{(n+2)(2x+1)^{n+1}} \cdot \frac{(n+1)(2x+1)^n}{nx^n}\right| = \lim_{n\to\infty} \left|\frac{x}{2x+1} \cdot \frac{(n+1)^2}{n(n+2)}\right| = \left|\frac{x}{2x+1}\right| < 1$
$\Rightarrow |x| < |2x+1|$; if $x > 0$, $|x| < |2x+1| \Rightarrow x < 2x+1 \Rightarrow x > -1$; if $-\frac{1}{2} < x < 0$, $|x| < |2x+1|$
$\Rightarrow -x < 2x+1 \Rightarrow 3x > -1 \Rightarrow x > -\frac{1}{3}$; if $x < -\frac{1}{2}$, $|x| < |2x+1| \Rightarrow -x < -2x-1 \Rightarrow x < -1$. Therefore, the series converges absolutely for $x < -1$ and $x > -\frac{1}{3}$.

19. (a) No, the limit does not appear to depend on the value of the constant a
(b) Yes, the limit depends on the value of b
(c) $s = \left(1 - \frac{\cos\left(\frac{a}{n}\right)}{n}\right)^n \Rightarrow \ln s = \frac{\ln\left(1 - \frac{\cos\left(\frac{a}{n}\right)}{n}\right)}{\left(\frac{1}{n}\right)} \Rightarrow \lim_{n\to\infty} \ln s = \frac{\left(\frac{1}{1-\frac{\cos\left(\frac{a}{n}\right)}{n}}\right)\left(\frac{-\frac{a}{n}\sin\left(\frac{a}{n}\right) + \cos\left(\frac{a}{n}\right)}{n^2}\right)}{\left(-\frac{1}{n^2}\right)}$
$= \lim_{n\to\infty} \frac{\frac{a}{n}\sin\left(\frac{a}{n}\right) - \cos\left(\frac{a}{n}\right)}{1 - \frac{\cos\left(\frac{a}{n}\right)}{n}} = \frac{0-1}{1-0} = -1 \Rightarrow \lim_{n\to\infty} s = e^{-1} \approx 0.3678794412$; similarly,
$\lim_{n\to\infty} \left(1 - \frac{\cos\left(\frac{a}{n}\right)}{bn}\right)^n = e^{-1/b}$

20. $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$; $\lim_{n\to\infty} \left[\left(\frac{1+\sin a_n}{2}\right)^n\right]^{1/n} = \lim_{n\to\infty} \left(\frac{1+\sin a_n}{2}\right) = \frac{1+\sin\left(\lim_{n\to\infty} a_n\right)}{2} = \frac{1+\sin 0}{2}$
$= \frac{1}{2} \Rightarrow$ the series converges by the nth-Root Test

21. $\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n \to \infty} \left|\frac{b^{n+1}x^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{b^n x^n}\right| < 1 \Rightarrow |bx| < 1 \Rightarrow -\frac{1}{b} < x < \frac{1}{b} = 5 \Rightarrow b = \pm\frac{1}{5}$

22. A polynomial has only a finite number of nonzero terms in its Taylor series, but the functions sin x, ln x and e^x have infinitely many nonzero terms in their Taylor expansions.

23. $\lim_{x \to 0} \frac{\sin(ax) - \sin x - x}{x^3} = \lim_{x \to 0} \frac{\left(ax - \frac{a^3x^3}{3!} + \dots\right) - \left(x - \frac{x^3}{3!} + \dots\right) - x}{x^3} = \lim_{x \to 0} \left[\frac{a-2}{x^2} - \frac{a^3}{3!} + \frac{1}{3!} - \left(\frac{a^5}{5!} - \frac{1}{5!}\right)x^2 + \dots\right]$ is finite if $a - 2 = 0 \Rightarrow a = 2$; $\lim_{x \to 0} \frac{\sin 2x - \sin x - x}{x^3} = -\frac{2^3}{3!} + \frac{1}{3!} = -\frac{7}{6}$

24. $\lim_{x \to 0} \frac{\cos ax - b}{2x^2} = -1 \Rightarrow \lim_{x \to 0} \frac{\left(1 - \frac{a^2x^2}{2} + \frac{a^4x^4}{4!} - \dots\right) - b}{2x^2} = -1 \Rightarrow \lim_{x \to 0} \left(\frac{1-b}{2x^2} - \frac{a^2}{4} + \frac{a^2x^2}{48} - \dots\right) = -1$ $\Rightarrow b = 1$ and $a = \pm 2$

25. (a) $\frac{u_n}{u_{n+1}} = \frac{(n+1)^2}{n^2} = 1 + \frac{2}{n} + \frac{1}{n^2} \Rightarrow C = 2 > 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

(b) $\frac{u_n}{u_{n+1}} = \frac{n+1}{n} = 1 + \frac{1}{n} + \frac{0}{n^2} \Rightarrow C = 1 \le 1$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

26. $\frac{u_n}{u_{n+1}} = \frac{2n(2n+1)}{(2n-1)^2} = \frac{4n^2+2n}{4n^2-4n+1} = 1 + \frac{\left(\frac{6}{4}\right)}{n} + \frac{5}{4n^2-4n+1} = 1 + \frac{\left(\frac{3}{2}\right)}{n} + \frac{\left[\frac{5n^2}{(4n^2-4n+1)}\right]}{n^2}$ after long division $\Rightarrow C = \frac{3}{2} > 1$ and $|f(n)| = \frac{5n^2}{4n^2-4n+1} = \frac{5}{\left(4 - \frac{4}{n} + \frac{1}{n^2}\right)} \le 5 \Rightarrow \sum_{n=1}^{\infty} u_n$ converges by Raabe's Test

27. (a) $\sum_{n=1}^{\infty} a_n = L \Rightarrow a_n^2 \le a_n \sum_{n=1}^{\infty} a_n = a_n L \Rightarrow \sum_{n=1}^{\infty} a_n^2$ converges by the Direct Comparison Test

(b) converges by the Limit Comparison Test: $\lim_{n \to \infty} \frac{\left(\frac{a_n}{1-a_n}\right)}{a_n} = \lim_{n \to \infty} \frac{1}{1-a_n} = 1$ since $\sum_{n=1}^{\infty} a_n$ converges and therefore $\lim_{x \to \infty} a_n = 0$

28. If $0 < a_n < 1$ then $|\ln(1-a_n)| = -\ln(1-a_n) = a_n + \frac{a_n^2}{2} + \frac{a_n^3}{3} + \dots < a_n + a_n^2 + a_n^3 + \dots = \frac{a_n}{1-a_n}$, a positive term of a convergent series, by the Limit Comparison Test and Exercise 27b

29. $(1-x)^{-1} = 1 + \sum_{n=1}^{\infty} x^n$ where $|x| < 1 \Rightarrow \frac{1}{(1-x)^2} = \frac{d}{dx}(1-x)^{-1} = \sum_{n=1}^{\infty} nx^{n-1}$ and when $x = \frac{1}{2}$ we have $4 = 1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \dots + n\left(\frac{1}{2}\right)^{n-1} + \dots$

30. (a) $\sum_{n=1}^{\infty} x^{n+1} = \frac{x^2}{1-x} \Rightarrow \sum_{n=1}^{\infty} (n+1)x^n = \frac{2x-x^2}{(1-x)^2} \Rightarrow \sum_{n=1}^{\infty} n(n+1)x^{n-1} = \frac{2}{(1-x)^3} \Rightarrow \sum_{n=1}^{\infty} n(n+1)x^n = \frac{2x}{(1-x)^3}$ $\Rightarrow \sum_{n=1}^{\infty} \frac{n(n+1)}{x^n} = \frac{\frac{2}{x}}{\left(1 - \frac{1}{x}\right)^3} = \frac{2x^2}{(x-1)^3}$, $|x| > 1$

(b) $x = \sum_{n=1}^{\infty} \frac{n(n+1)}{x^n} \Rightarrow x = \frac{2x^2}{(x-1)^3} \Rightarrow x^3 - 3x^2 + x - 1 = 0 \Rightarrow x = 1 + \left(1 + \frac{\sqrt{57}}{9}\right)^{1/3} + \left(1 - \frac{\sqrt{57}}{9}\right)^{1/3}$ ≈ 2.769292, using a CAS or calculator

31. (a) $\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(1 + x + x^2 + x^3 + \dots\right) = 1 + 2x + 3x^2 + 4x^3 + \dots = \sum_{n=1}^{\infty} nx^{n-1}$

(b) from part (a) we have $\sum_{n=1}^{\infty} n\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right) = \left(\frac{1}{6}\right)\left[\frac{1}{1-\left(\frac{5}{6}\right)}\right]^2 = 6$

(c) from part (a) we have $\sum_{n=1}^{\infty} np^{n-1}q = \frac{q}{(1-p)^2} = \frac{q}{q^2} = \frac{1}{q}$

32. (a) $\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} 2^{-k} = \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)} = 1$ and $E(x) = \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} k2^{-k} = \frac{1}{2} \sum_{k=1}^{\infty} k2^{1-k} = \left(\frac{1}{2}\right) \frac{1}{\left[1-\left(\frac{1}{2}\right)\right]^2} = 2$ by Exercise 31(a)

(b) $\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} \frac{5^{k-1}}{6^k} = \frac{1}{5} \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^k = \left(\frac{1}{5}\right) \left[\frac{\left(\frac{5}{6}\right)}{1-\left(\frac{5}{6}\right)}\right] = 1$ and $E(x) = \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} k \frac{5^{k-1}}{6^k} = \frac{1}{6} \sum_{k=1}^{\infty} k \left(\frac{5}{6}\right)^{k-1}$
$= \left(\frac{1}{6}\right) \frac{1}{\left[1-\left(\frac{5}{6}\right)\right]^2} = 6$

(c) $\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{k \to \infty} \left(1 - \frac{1}{k+1}\right) = 1$ and $E(x) = \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} k \left(\frac{1}{k(k+1)}\right)$
$= \sum_{k=1}^{\infty} \frac{1}{k+1}$, a divergent series so that $E(x)$ does not exist

33. (a) $R_n = C_0 e^{-kt_0} + C_0 e^{-2kt_0} + \ldots + C_0 e^{-nkt_0} = \frac{C_0 e^{-kt_0}\left(1-e^{-nkt_0}\right)}{1-e^{-kt_0}} \Rightarrow R = \lim_{n \to \infty} R_n = \frac{C_0 e^{-kt_0}}{1-e^{-kt_0}} = \frac{C_0}{e^{kt_0}-1}$

(b) $R_n = \frac{e^{-1}\left(1-e^{-n}\right)}{1-e^{-1}} \Rightarrow R_1 = e^{-1} \approx 0.36787944$ and $R_{10} = \frac{e^{-1}\left(1-e^{-10}\right)}{1-e^{-1}} \approx 0.58195028$;
$R = \frac{1}{e-1} \approx 0.58197671$; $R - R_{10} \approx 0.00002643 \Rightarrow \frac{R-R_{10}}{R} < 0.0001$

(c) $R_n = \frac{e^{-.1}\left(1-e^{-.1n}\right)}{1-e^{-.1}}$, $\frac{R}{2} = \frac{1}{2}\left(\frac{1}{e^{.1}-1}\right) \approx 4.7541659$; $R_n > \frac{R}{2} \Rightarrow \frac{1-e^{-.1n}}{e^{.1}-1} > \left(\frac{1}{2}\right)\left(\frac{1}{e^{.1}-1}\right)$
$\Rightarrow 1 - e^{-n/10} > \frac{1}{2} \Rightarrow e^{-n/10} < \frac{1}{2} \Rightarrow -\frac{n}{10} < \ln\left(\frac{1}{2}\right) \Rightarrow \frac{n}{10} > -\ln\left(\frac{1}{2}\right) \Rightarrow n > 6.93 \Rightarrow n = 7$

34. (a) $R = \frac{C_0}{e^{kt_0}-1} \Rightarrow Re^{kt_0} = R + C_0 = C_H \Rightarrow e^{kt_0} = \frac{C_H}{C_L} \Rightarrow t_0 = \frac{1}{k} \ln\left(\frac{C_H}{C_L}\right)$

(b) $t_0 = \frac{1}{0.05} \ln e = 20$ hrs

(c) Give an initial dose that produces a concentration of 2 mg/ml followed every $t_0 = \frac{1}{0.02} \ln\left(\frac{2}{0.5}\right) \approx 69.31$ hrs by a dose that raises the concentration by 1.5 mg/ml

(d) $t_0 = \frac{1}{0.2} \ln\left(\frac{0.1}{0.03}\right) = 5 \ln\left(\frac{10}{3}\right) \approx 6$ hrs

NOTES: